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Technical Note 167

THE ERROR RATES IN MULTIPLE FSK SYSTEMS AND THE SIGNAL-TO-NOISE CHARACTERISTICS OF FM AND PCM-FS SYSTEMS

HIROSHI AKIMA

U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS

THE NATIONAL BUREAU OF STANDARDS

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CONTENTS

The Error Rates in Multiple FSK Systems and the Signal-to-Noise Characteristics of FM and PCM-FS Systems

Hiroshi Akima

The element and symbol error rates in multiple FSK (frequency-shift-keying) systems and the output SNR (signal-to-noise ratio) in FM (frequency-modulation) and PCM-FS (pulse-code-modulation-frequency-shift) systems are evaluated for wide ranges of system parameters, assuming that the incoming signal and noise in the demodulator are a fading-free signal and an additive white Gaussian noise, respectively. It is shown that the required intrinsic SNR for an assigned value of symbol error rate in multiple FSK systems can be reduced by increasing the number of frequencies in the keying. The possibility of improving the threshold of FM systems beyond that of conventional ones by modulating the carrier with sampled values and demodulating the modulated wave with a band-dividing demodulator is shown. The value of the intrinsic SNR at the threshold increases with the value of modulation index in band-dividing FM systems, and with the number of quantizing levels in PCM-FS systems when the base in the coding or the number of digits for each sample is kept constant. The maximum output SNR in PCM-FS systems depends only on the number of quantizing levels and not on the base, whereas the threshold decreases as the base increases. From the comparison of the threshold in band-dividing FM systems with that in PCM-FS systems it is shown that the latter cannot be lower than the former but can only approach the former when the base approaches the number of quantizing levels. Brief discussions on the threshold effects in frequency-lock and phase-lock FM demodulators suggest that the threshold of these feedback FM demodulators cannot be improved beyond that of a band-dividing one.

1. INTRODUCTION

In radio communication either the amplitude, the frequency, or the phase of a sinusoidal wave of radio frequency (carrier) is modulated by a modulating signal. A modulation of frequency of a carrier is called frequency-modulation (FM). In general, the modulating signal can be classified into two categories, i.e., analog and digital, and when the modulating signal is digital, the frequency modulation is called frequency-shift-keying (FSK or FS).

The information signal to be transmitted can also be classified into the same two categories. The quality of the signal at the final destination can be expressed by the output signal-to-noise ratio (SNR) and the symbol (or character) error rate corresponding to the analog and digital information signal, respectively. The distinction between the classification of the modulating signal and that of the information signal must be stressed. In a pulse-code-modulation (PCM) system, for example, the information signal is analog, although the modulating signal is digital. The quality of the received signal in this system, therefore, should be discussed with the output SNR.

One of the most prominent features of FM systems is their signal-to-noise improving characteristic over amplitude-modulation (AM) systems, as demonstrated by Armstrong [1936]. Many studies have been carried out to determine the signal-to-noise characteristics of FM systems. It has been clarified that the system has noise-suppressing characteristics

providing the SNR at the input of the demodulator is equal to or larger than the threshold value of approximately 10 decibels. It has also been shown that the output SNR decreases rapidly as the input SNR decreases beyond the threshold value [Crosby, 1937; Stumpers, 1948].[1]

This value of threshold, however, is valid only when we use a conventional FM demodulator which is composed of an amplitude limiter, a frequency discriminator, and a low-pass filter. Several demodulators have been suggested in order to improve the threshold of FM systems. A frequency-lock demodulator, suggested by Chaffee [1939], was used in the Project Echo satellite communication system, and yielded a considerable amount of threshold improvement [Ruthroff, 1961]. On the other hand, phase-lock demodulators, suggested by Lehan and Parks [1953] as practical approaches to the optimum FM demodulator, have been used in satellite tracking and telemetering systems. A phase-lock FM demodulator was also developed for microwave telephone multiplex communication channels [Morita and Ito, 1960]. The threshold of the frequency-lock and phase-lock demodulator has been studied by many [Jaffe and Rechtin, 1955; Margolis, 1957; Gilchriest, 1958; Weaver, 1959; Martin, 1960; Choate, 1960; Spilker, 1961; Enloe, 1962; etc.], but to the author's knowledge, the ultimate limit of improving the threshold has not yet been determined.

An important idea for analyzing the signal-to-noise characteristics of FM systems was suggested by Lehan [1954]. He proposed the conceptual idea of <u>dividing</u> the receiver bandwidth into many channels (the bandwidth of each channel being equal to twice the maximum frequency of the information signal), <u>measuring</u> the amplitude at each channel, <u>selecting</u> the channel with maximum amplitude as a signal channel from amplitude comparisons, and <u>assuming</u> that the frequency is measured by the center frequency of the signal channel. In his paper the transmitted wave is assumed to be frequency-modulated by a discrete signal, which coincides with the original information signal at every sampling point, equally spaced by a Nyquist interval. He also suggested a phase-lock FM demodulator as a practical approach to his band-dividing one. This band-dividing idea was further developed, and the signal-to-noise characteristics of an FM system with a band-dividing demodulator were determined [Akima, 1961, 1963; Battail, 1962]. The above studies also suggest the similarity of FM systems to PCM-FS systems. The PCM system was invented by Reeves [1939; 1942], and the superiority of the system was demonstrated by Goodall [1947]. The basic characteristics of the system were discussed by Oliver, et. al. [1948]. The essential feature of the PCM system is <u>sampling</u> at every sampling point, <u>quantizing</u> the sampled value into L quantizing levels, and <u>coding</u> the quantized value with the base in the coding N. The system uses n elements for every sampled value, where n is equal to $\log_N L$. When the coded signal is used to frequency-modulate the carrier, the system is called PCM-FS system.

Although PCM systems were originated with the base N = 2, a better system can be obtained if N is increased, as shown by Billings [1958] and Viterbi [1962].

Consider that the sampled value from an analog information signal at every sampling point, equally spaced by a Nyquist interval, is quantized into L levels, and that this quantized signal is used to frequency-modulate the carrier. Then the modulated wave is the same as the one in a multiple FSK system. If the number of the quantizing levels L is very large, the modulated wave can be considered to yield approximately the same amount of information as the continuously frequency-modulated wave by the original information signal. On the other hand, the discrete modulation can also be considered to be a special case of PCM-FS with the base in the coding N equal to the number of quantizing levels L. The above observations, therefore, indicate the necessity of constructing a general theory which covers the signal-to-noise characteristics of both FM and PCM-FS systems.

It is clear from these observations that the starting point of this theory is to analyze the error rates in multiple FSK systems. The superiority of the multiple FSK system to binary ones was shown by Jordan, et. al. [1955],and Robin and Murray [1958] experimentally.

[1] Figures in brackets indicate the literature references on page 40.

Although the element error rates in the multiple FSK systems have been studied theoretically [Reiger, 1958; Turin, 1958; Helstrom, 1960; Viterbi, 1962], the symbol (or character) error rates have not yet been studied satisfactorily.*

In this paper the error studies on multiple FSK systems are extended and, based on these studies, the output SNR in FM and PCM-FS systems are calculated for wide ranges of system parameters. The signal-to-noise characteristics of these systems are compared with each other.

There are several methods of comparing these characteristics of communication systems [Jelonek, 1952; Beard and Wheeldon, 1960; Helstrom, 1960; Lieberman, 1961]. In this paper, however, these characteristics are discussed from the standpoint of communication system engineering. For this purpose the quality of the signal at the final destination is expressed by the symbol error rate and the output SNR corresponding to the digital and analog information signal, respectively, and the concepts of the intrinsic bandwidth and intrinsic SNR [de Jager and Greefkes, 1957] are effectively used in representing the characteristics.

In order to analyze the basic characteristics of the systems it is assumed throughout this paper that the incoming signal in the demodulator is a fading-free signal, and that the incoming noise is an additive Gaussian noise with a flat spectrum across the bandwidth of interest.

2. THE ERROR RATES IN MULTIPLE FSK SYSTEMS

The element error rate p_e in multiple FSK systems with coherent detectors is given by

$$p_e = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{(v-v_s)^2}{2}\right)\left\{1 - \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{v} \exp\left(-\frac{u^2}{2}\right) du\right]^{N-1}\right\} dv, \quad (1)$$

where N is the number of frequencies in the keying and v_s is the normalized amplitude of the incoming signal voltage with the effective value of the noise voltage in each channel as a unit [Helstrom, 1960; Lieberman, 1961]. If we define the channel SNR R_c as the ratio of the incoming signal power to the incoming noise power in each channel, the relation between v_s and R_c is given by

$$R_c = v_s^2/2. \quad (2)$$

In an ideal case where the bandwidth of each channel B_c can be equal to the reciprocal of the unit time duration of each digit, R_c coincides with the alternative expression of signal energy

* A digital information signal is composed of a sequence of symbols (or characters), and each symbol (or character) is transmitted as a sequence of elements.

in each digit per noise power density. The values of p_e are calculated by an electronic computer as a function of N and R_c, and shown in figure 1.

On the other hand, the element error rate p_e in multiple FSK systems with incoherent detectors is given by

$$p_e = \int_0^{\infty} v \exp\left(-\frac{v^2+v_s^2}{2}\right) I_0(v_s v)\left\{1 - \left[1 - \exp\left(-\frac{v^2}{2}\right)\right]^{N-1}\right\} dv \tag{3}$$

or by its equivalent

$$p_e = \frac{1}{N}\left[\sum_{k=2}^{N} (-1)^k \binom{N}{k} \exp\left(-\frac{(k-1)R_c}{k}\right)\right], \tag{4}$$

where N and v_s are the same as before, and the function $I_0(x)$ is a modified Bessel function of the 1st kind of the 0th order, and $\binom{N}{k}$ is the number of combinations of k out of N [Reiger, 1958; Lieberman, 1961]. For relatively small values of N, (4) can be used. For large values of N, however, the numerical integration of (3) is more convenient than using (4). The values of p_e are calculated by an electronic computer as a function of N and R_c, and shown in figure 2.

In either case of coherent or incoherent detectors the element error rate p_e approaches (N-1)/N when R_c tends to zero. For large value of R_c, on the other hand, p_e can be approximated by (N-1) times that in a binary system [Helstrom, 1960].

Some of the curves in figures 1 and 2 are compared with each other in figure 3. It is clear from figure 3 that the coherent detection is always better than the incoherent one but the difference between the two detections decreases as the number of frequencies N increases. Because of the difficulties often encountered in practical implementations, we shall continue our present study only on the incoherent detection.

Next we shall calculate the symbol (or character) error rate in multiple FSK systems. If the information signal is an L-alphabet system, or if it consists of L symbols (or characters), and when it is transmitted over an FSK link with N frequencies, the number of elements n for each symbol is related to L and N by

$$L = N^n. \tag{5}$$

As each element error occurs independently under the conditions assumed in this paper, the symbol error rate p_s can be obtained by

$$p_s = 1 - (1 - p_e)^n. \tag{6}$$

To study the system performances we shall introduce the concepts of the intrinsic bandwidth and the intrinsic SNR. The intrinsic bandwidth is a bandwidth which is intrinsic to the information signal. Although there is some arbitrariness in defining the intrinsic bandwidth, it is convenient in digital systems to take a binary system as a reference system

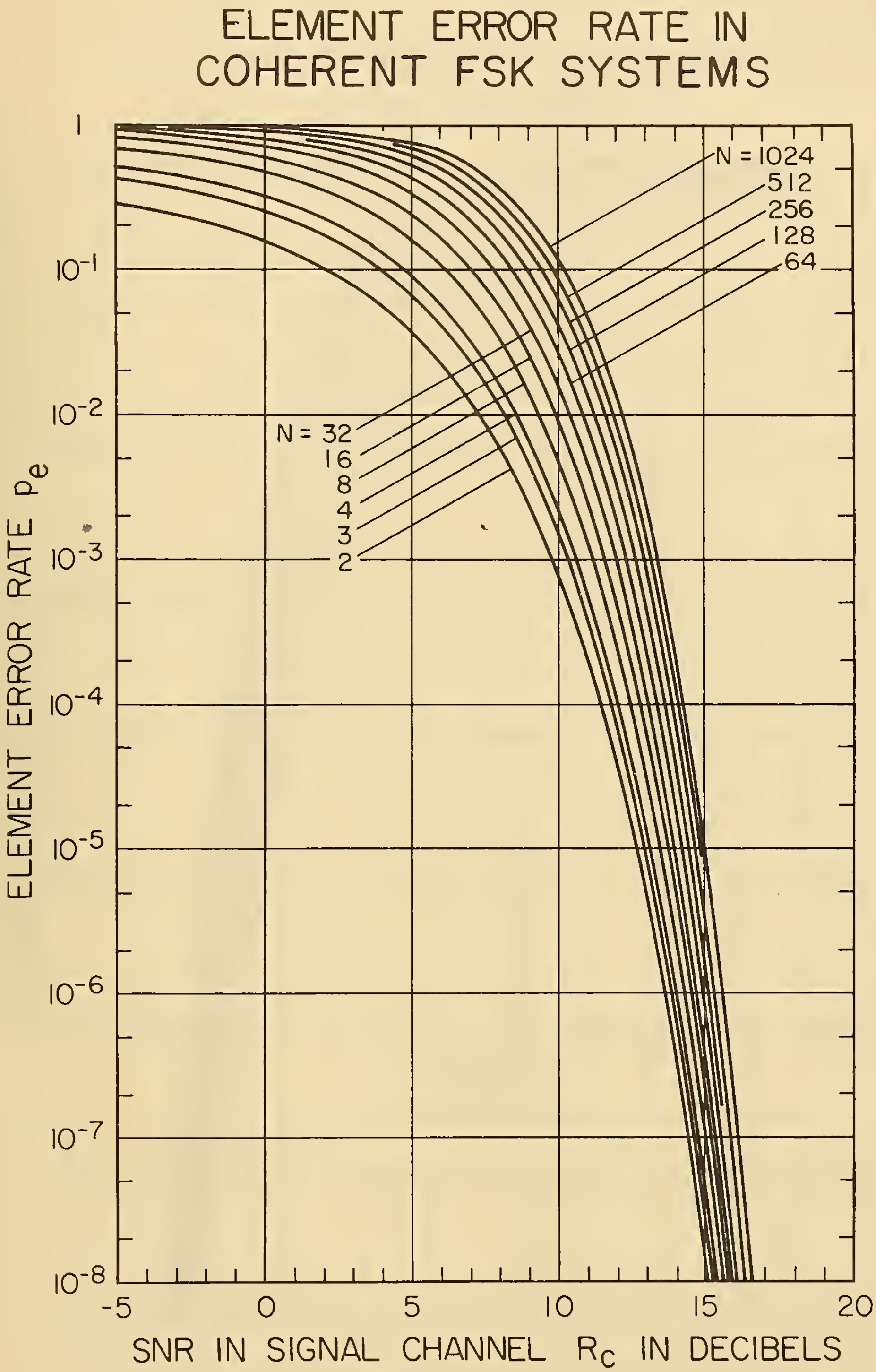

ELEMENT ERROR RATE IN
COHERENT FSK SYSTEMS
N = 1024
512
256
128
64
N = 32
16
8
4
3
2
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
ELEMENT ERROR RATE P_e
-5
0
5
10
15
20
SNR IN SIGNAL CHANNEL R_c IN DECIBELS

Figure 1

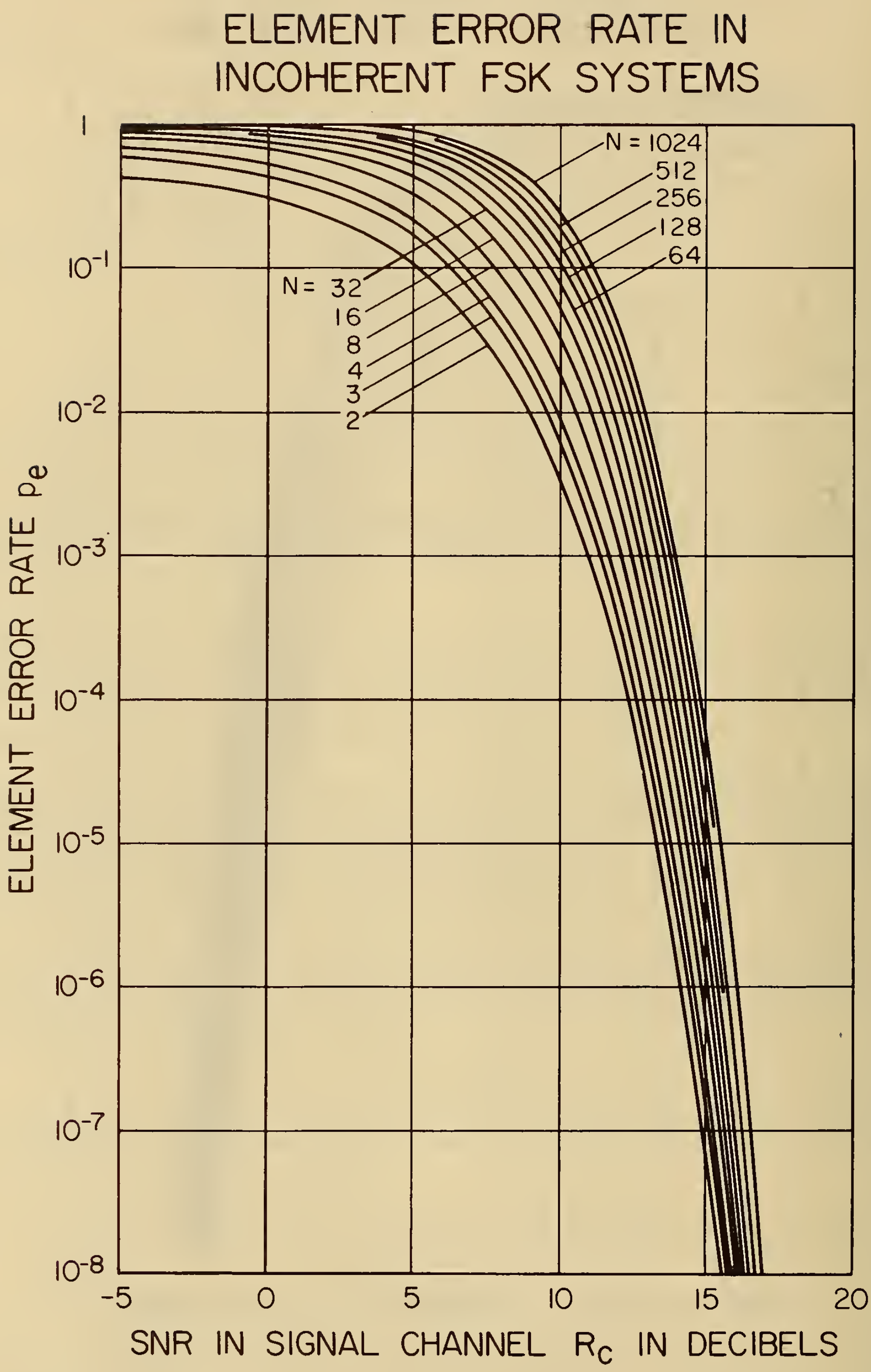

ELEMENT ERROR RATE IN
INCOHERENT FSK SYSTEMS
N = 1024
512
256
128
64
N = 32
16
8
4
3
2
1
10^-1
10^-2
10^-3
10^-4
10^-5
10^-6
10^-7
10^-8
ELEMENT ERROR RATE pe
-5
0
5
10
15
20
SNR IN SIGNAL CHANNEL Rc IN DECIBELS

Figure 2

A COMPARISON OF ELEMENT ERROR RATE BETWEEN COHERENT AND INCOHERENT FSK SYSTEMS

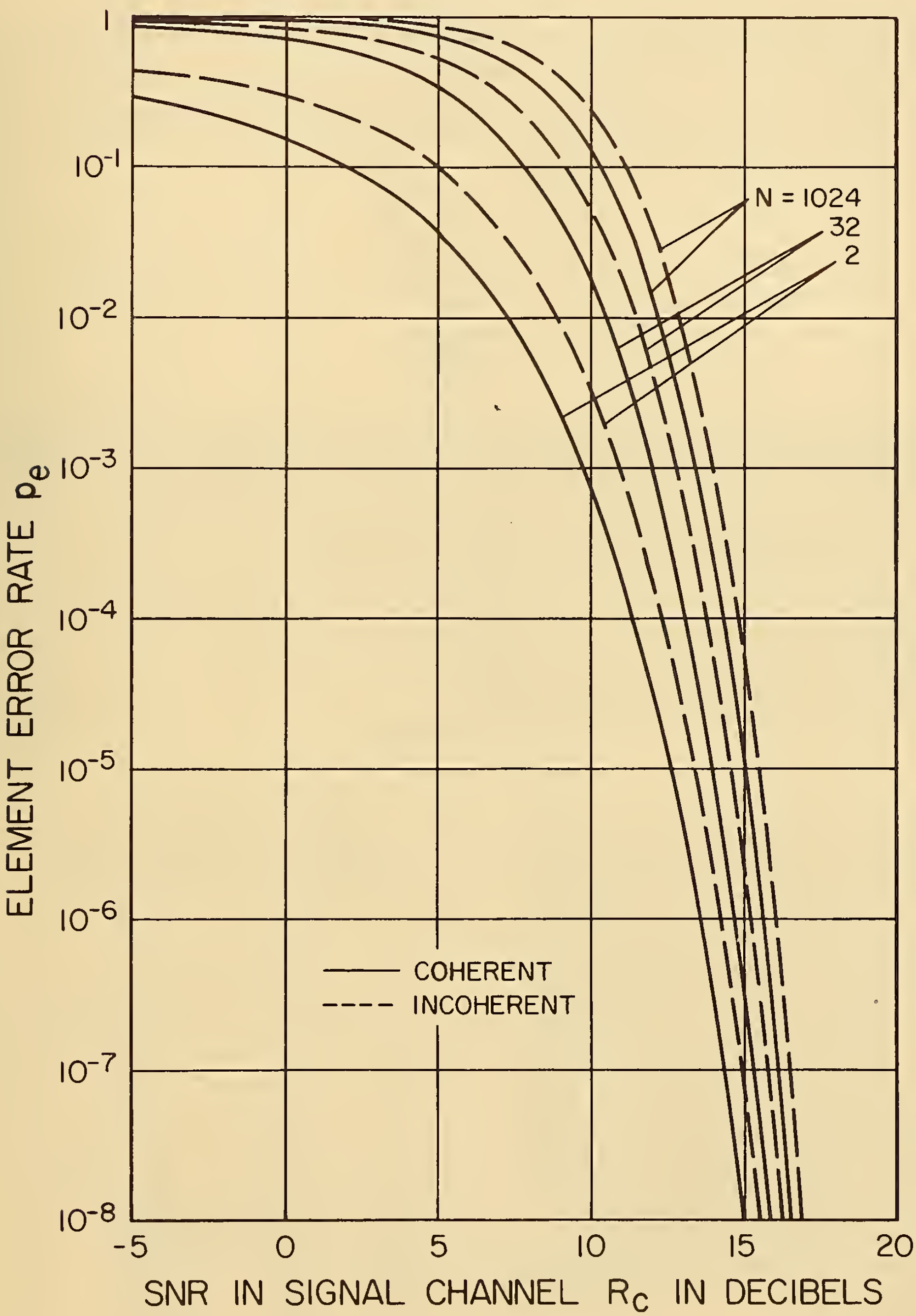


Figure 3

and to take the reciprocal of the duration of each element in the binary system as the intrinsic bandwidth B_i. The value of B_i expressed in cycles/second is equal to the value of the transmission rate expressed in bits/second. Then the intrinsic SNR R_i in digital systems can be defined as the ratio of the incoming signal power to the incoming noise power contained in a band of width B_i.

When the number of frequencies N is increased in FSK systems, the duration of each element can be increased to $\log_2 N$ times that in binary systems in order to transmit an equal amount of information. The channel bandwidth B_c can, therefore, be reduced by a factor $\log_2 N$. As the intrinsic bandwidth B_i is equal to the channel bandwidth B_c in ideal binary FSK systems, we obtain the relations

$$B_c = B_i / \log_2 N \tag{7}$$

and

$$R_i = R_c / \log_2 N. \tag{8}$$

The symbol error rates for some values of L and N are calculated from the element error rates given above, and shown in figures 4, 5, and 6 as a function of the intrinsic SNR R_i. It is clear from these figures that the required intrinsic SNR (or the required signal power) for an assigned value of symbol error rate in multiple FSK systems can be reduced by increasing the number of frequencies in the keying.

We shall give a brief comment on band occupancy in multiple FSK systems. As the overall bandwidth B_a is equal to NB_c, and as the channel bandwidth B_c is related to the intrinsic bandwidth B_i by (7), we obtain the relation

$$B_a / B_i = N / \log_2 N. \tag{9}$$

The values of the bandwidth ratio B_a/B_i for some integers of N are given in Table 1. It is clear from the table that the ratio B_a/B_i decreases at first, but it increases after passing its minimum, as N increases. (The minimum takes place at N = e = 2.7183, and the minimum value is $e/\log_2 e = 1.8842$.) It is important to notice that the ternary system requires a narrower bandwidth than the binary one and the quaternary system requires the same bandwidth as the binary one, and these systems achieve a reduction of the required intrinsic SNR. A greater reduction of the required intrinsic SNR by increasing the number of frequencies N beyond 4, however, must be accompanied by an increase of the bandwidth ratio B_a/B_i.

Table 1. Bandwidth ratio B_a/B_i in N-ary FSK systems.

N	B_a/B_i
2	2
3	1.893
4	2
5	2.153
6	2.321
8	2.667
16	4
32	6.4
64	10.67
128	18.29
256	32
512	56.89
1024	102.4

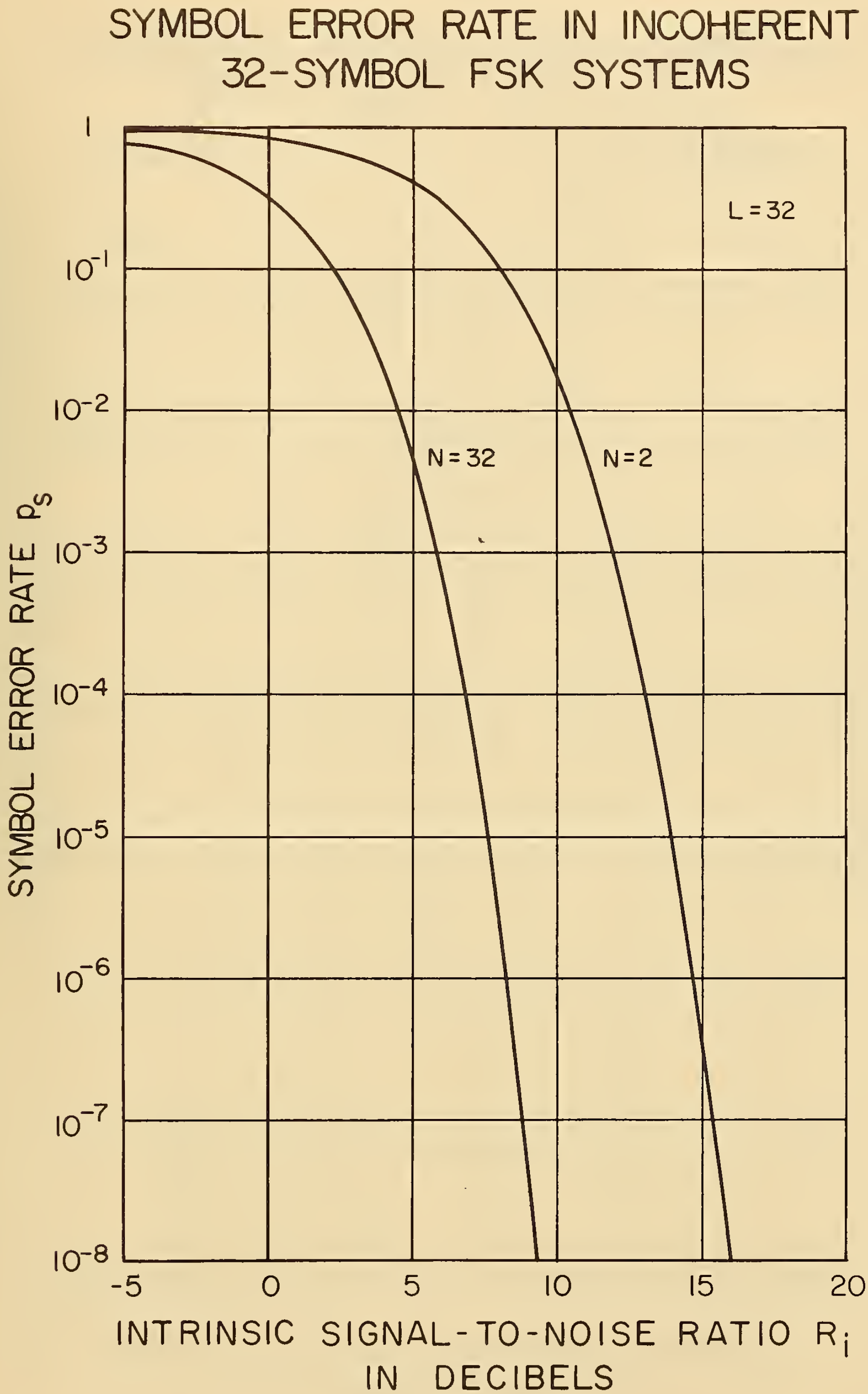

SYMBOL ERROR RATE IN INCOHERENT
32-SYMBOL FSK SYSTEMS
L = 32
N = 32
N = 2
SYMBOL ERROR RATE p_s
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
-5
0
5
10
15
20
INTRINSIC SIGNAL-TO-NOISE RATIO R_i
IN DECIBELS

Figure 4

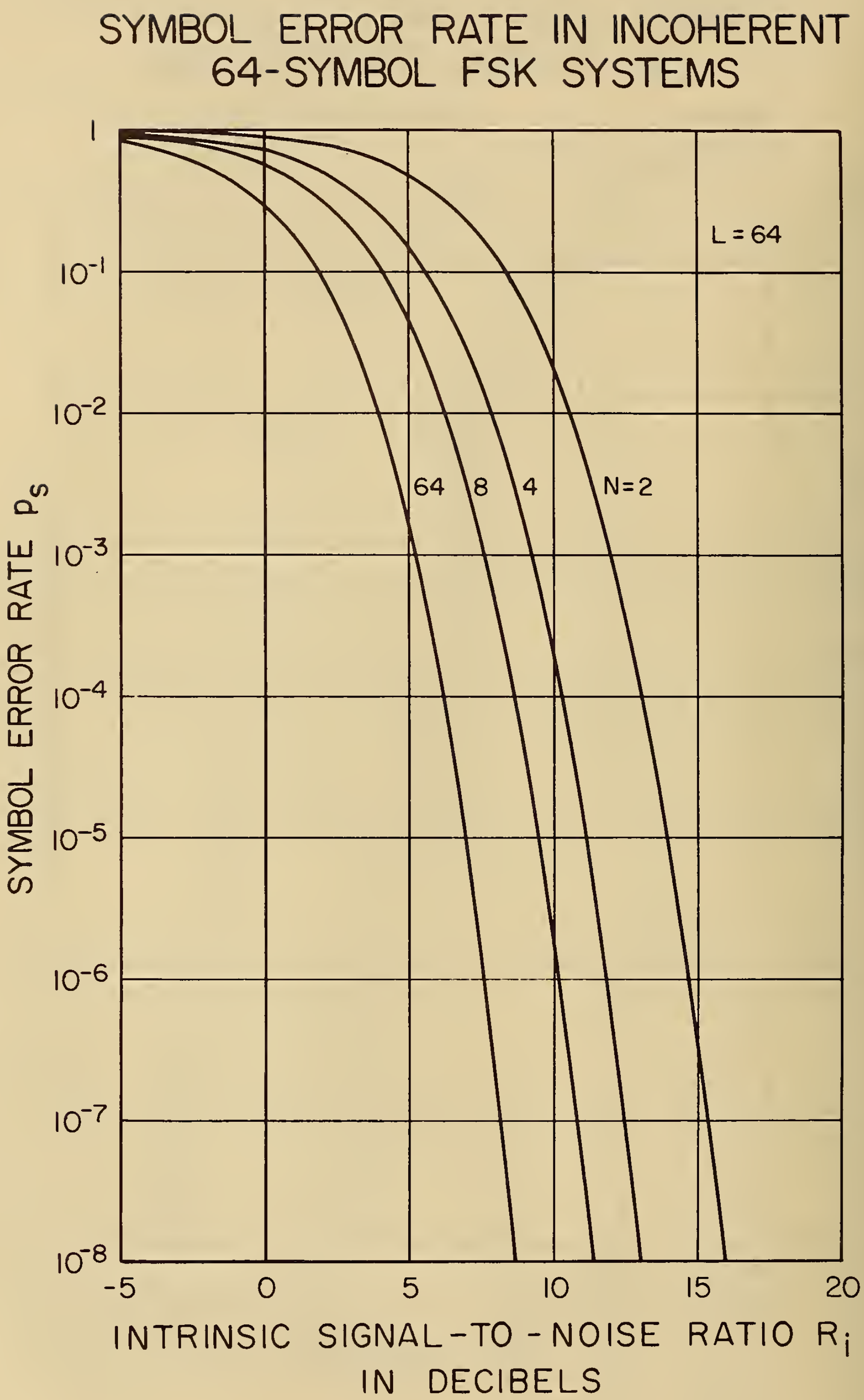

SYMBOL ERROR RATE IN INCOHERENT
64-SYMBOL FSK SYSTEMS
L = 64
64
8
4
N=2
SYMBOL ERROR RATE P_S
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
-5
0
5
10
15
20
INTRINSIC SIGNAL-TO-NOISE RATIO R_i
IN DECIBELS

Figure 5

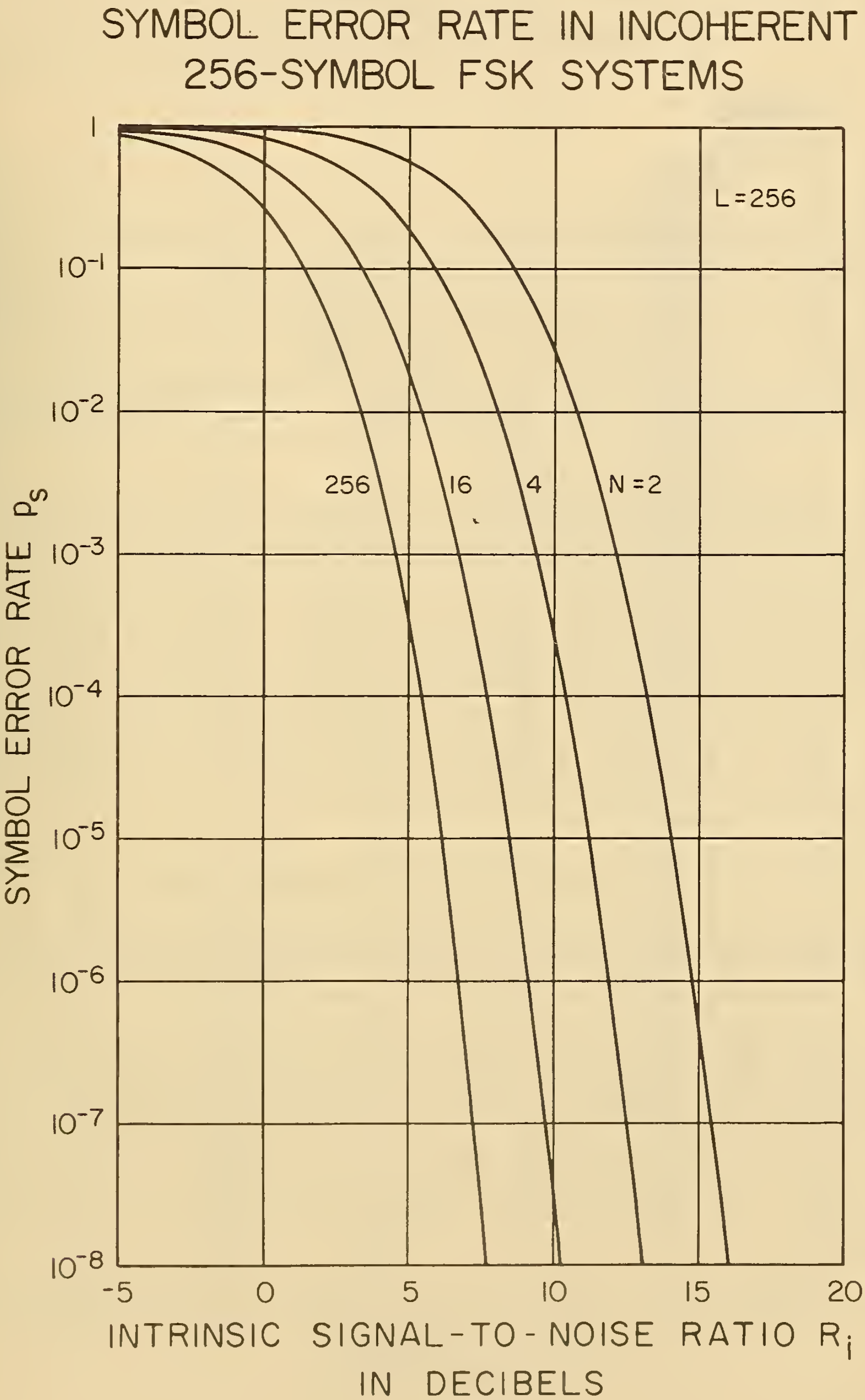

SYMBOL ERROR RATE IN INCOHERENT
256-SYMBOL FSK SYSTEMS
L=256
256
16
4
N=2
SYMBOL ERROR RATE P_S
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
-5
0
5
10
15
20
INTRINSIC SIGNAL-TO-NOISE RATIO R_i
IN DECIBELS

Figure 6

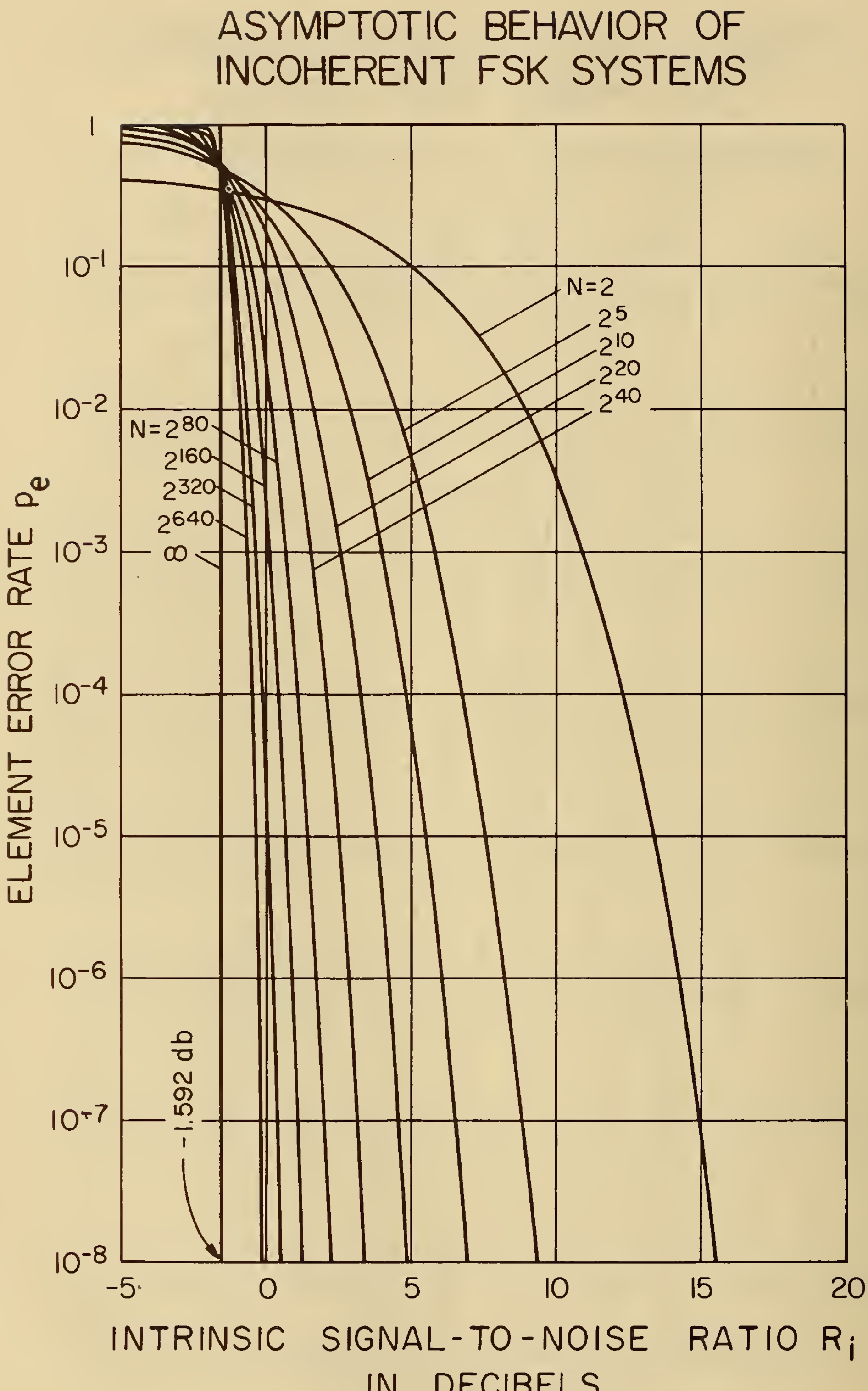

ASYMPTOTIC BEHAVIOR OF INCOHERENT FSK SYSTEMS
ELEMENT ERROR RATE p_e
INTRINSIC SIGNAL-TO-NOISE RATIO R_i IN DECIBELS
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
-5
0
5
10
15
20
N=2
2^{5}
2^{10}
2^{20}
2^{40}
N=2^{80}
2^{160}
2^{320}
2^{640}
∞
-1.592 db

Figure 7

So far we have studied a system in which one carrier is modulated or keyed by n elements for each symbol in time sequence. We can consider another system, in which n carriers are keyed simultaneously by frequency division. In the latter system the duration of each element can be increased to n times that in the former, and therefore, the channel bandwidth B_c can be reduced by a factor n, i. e.,

$$B_c = B_i / (n \log_2 N), \tag{10}$$

instead of (7). To keep the same error rate in these two systems the channel SNR R_c should be kept constant, because the element error rate p_e depends only on R_c and N. As the noise power in each channel is reduced by a factor n, the signal power of each carrier in the latter can be reduced by the same factor. The total signal power in the latter, however, should be the same as the signal power in the former, because n carriers are used in the latter. Moreover, the overall bandwidth B_a in the latter is the same as in the former, because we must employ n times N frequencies in the latter. It is clear from these observations that both (8) and (9) hold in the latter as well as in the former, and therefore, figures 4, 5, and 6 are valid in the latter, too.

It is interesting to observe how the multiple FSK systems behave when the number of frequencies N is increased. This behavior has already been studied by Turin [1959], but it will be studied here with a higher order approximation given in Appendix A. The relations between the element error rate p_e and the intrinsic SNR R_i are calculated by this approximation and are shown in figure 7. In the limit of infinite N, the relation becomes a vertical straight line at $R_i = \log_e 2$ (= - 1.592 db). It is shown how slowly this critical value of R_i can be approached by increasing N. This critical value of R_i corresponds to r = 1 in Turin's paper, and therefore, it coincides with the value which is obtained by letting the channel capacity be equal to the transmission rate in Shannon's channel-capacity theorem [Shannon, 1948].

3. SIGNAL-TO-NOISE CHARACTERISTICS OF FM SYSTEMS

In this section we shall discuss the signal-to-noise characteristics of a communication system in which a radio frequency carrier is frequency-modulated by an analog signal. This system is called the frequency-modulation (FM) system in its narrower sense.

In the transmission of an analog information signal it is convenient to take a single-sideband (SSB) system as a reference and to take the maximum frequency of the information signal f_m as the intrinsic bandwidth B_i, i. e., $B_i = f_m$. The intrinsic SNR R_i can be defined as the ratio of the incoming signal power to the incoming noise power contained in a band of width B_i, in exactly the same manner as in the transmission of a digital information signal. The quality of the signal at the final destination can be expressed in terms of the output SNR R_{out}.

The minimum overall bandwidth of an FM receiver B_a can be expressed by the well-known relation

$$B_a = 2(1 + m) f_m = 2(1 + m) B_i, \tag{11}$$

where

$$m = f_d / f_m = \text{modulation index} \tag{12}$$

and f_d is the maximum frequency deviation of the modulated wave.

In receiving a frequency-modulated signal there are several demodulating schemes. One of the most popular demodulators is a conventional FM demodulator, which consists of an amplitude limiter, a frequency discriminator, and a low-pass filter. A frequency discriminator can be replaced by a frequency counter, which counts the number of zero-crossings in short intervals.

The output SNR R_{out} in FM systems with conventional demodulators is proportional to the intrinsic SNR R_i and is given by

$$R_{out} = \frac{3}{2} m^2 R_i, \tag{13}$$

if the input SNR is larger than the threshold value of approximately 10 decibels. In terms of the intrinsic SNR the threshold can be expressed by

$$10 \log_{10} R_i = 13 + 10 \log_{10}(1 + m) \text{ db}, \tag{14}$$

as is clear from (11). The relations between R_i and R_{out} below the threshold are calculated from the data of output noise spectrum and modulation suppression ratio given by Stumpers [1948], and shown in figure 8. To calculate these relations the shape of IF pass-band is assumed to be rectangular and the noise spectrum in case of no modulation is used. From figure 8 it should be noticed that the system with the modulation index m = 2 requires less signal power to obtain a relatively low output SNR.

We can draw an envelope of the curves in figure 8, and regard it as the improvement limit of FM systems with conventional demodulators. As mentioned above, several techniques have been suggested to improve the characteristics of FM systems. The aim of these techniques can be considered to move the envelope toward the left. Then an important problem is raised on the limit of moving the envelope.

In order to solve the problem we assume for a while that the modulating signal is a discrete signal which coincides with the original information signal at every sampling point, equally spaced in time by the Nyquist interval corresponding to the maximum frequency of the information signal f_m. After Shannon's sampling theorem [Shannon, 1948] this discrete modulating signal can be considered to yield exactly the same information as the original information signal. Developing Lehan's band-dividing idea [Lehan, 1954] we can reach a new model of an FM demodulator, in which the incoming wave is divided into several channels. The simultaneous measurements of amplitude and frequency are made in each channel at the end of every Nyquist interval. The channel with a maximum amplitude is selected as the signal channel, and the output of the frequency-measuring circuit in the signal channel is sent to the output of the whole demodulator [Akima, 1961, 1963]. The bandwidth of each channel B_c is equal to twice the maximum frequency of the original information signal f_m, i.e.,

$$B_c = 2 f_m = 2 B_i. \tag{15}$$

The relation between the channel SNR R_c and the intrinsic SNR R_i, therefore, is given by

$$R_c = R_i/2. \tag{16}$$

SIGNAL-TO-NOISE CHARACTERISTICS OF FM SYSTEMS WITH CONVENTIONAL DEMODULATORS
(CALCULATED FROM NOISE DATA WITHOUT MODULATION)

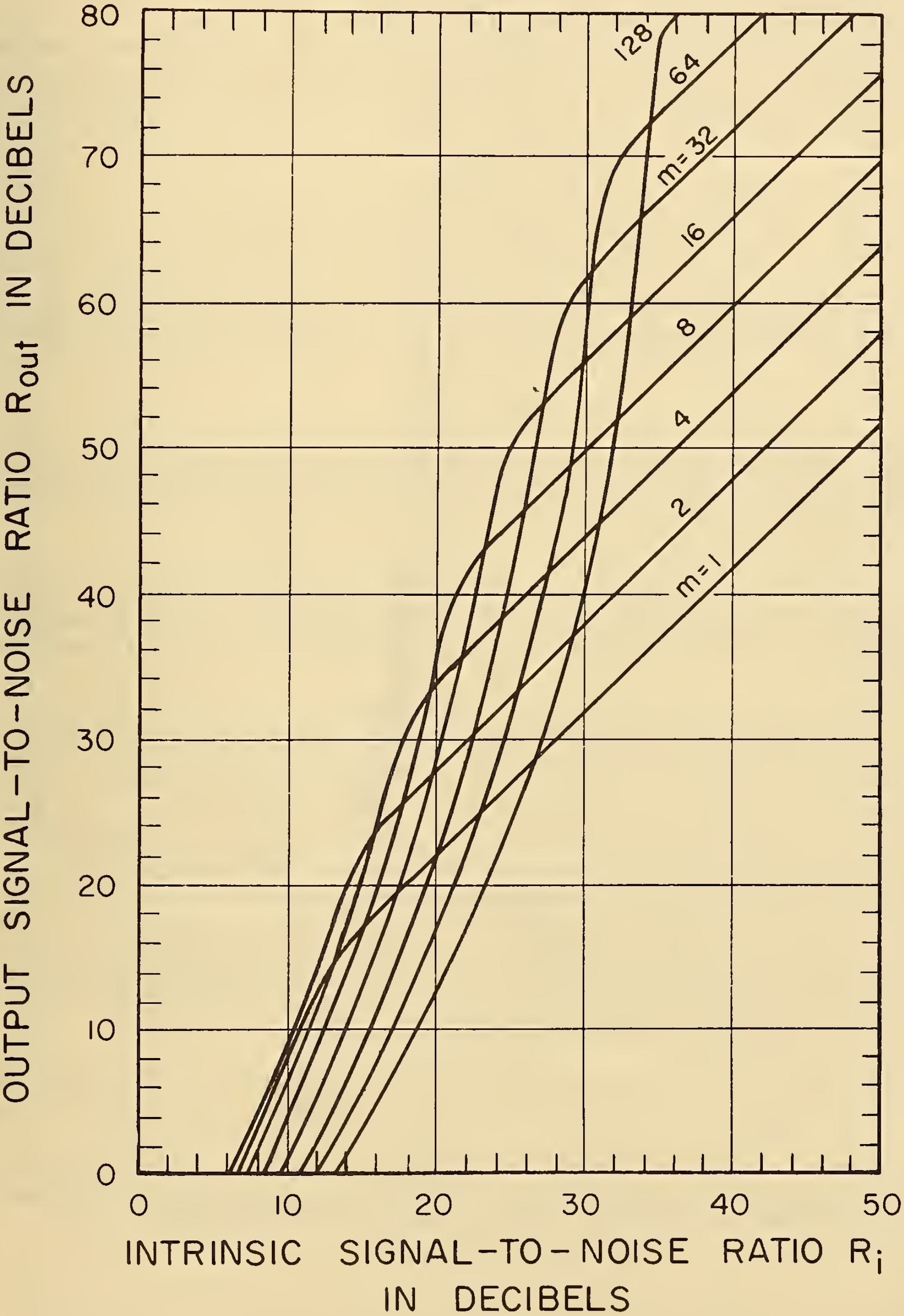


Figure 8

From (11) and (15) the number of channels N is given by

$$N = m + 1 \tag{17}$$

for an integer of modulation index m.

In our band-dividing demodulator the selection of the signal channel can make errors. If this should happen, an additional noise power will be produced at the output of the demodulator besides the output noise power due to the incoming noise in the signal channel, and the phenomenon of "modulation suppression" or "signal suppression" will also take place. In Appendix B it is shown that the output SNR R_{out} in the band-dividing demodulator is given by

$$R_{out} = \frac{3}{2} m^2 R_i \frac{[1 - (m + 1)p]^2}{1 + m(m + 1)(m + 2)[2 - (m + 1)p] pR_i}, \tag{18}$$

where p is the probability that any noise channel is selected as the signal channel by mistake. This probability p is related to the element error rate p_e in incoherent N-ary FSK systems by

$$p = p_e/(N - 1), \tag{19}$$

and it is, therefore, given as a function of the channel SNR R_c and the number of channels N. From the relations given above we can determine the relations between the intrinsic SNR R_i and the output SNR R_{out}.

When the input SNR is so large that the probability of mis-selection of the signal channel p is negligibly small, (18) coincides with (13). We can see, therefore, that the output SNR is the same in both the conventional and the band-dividing FM systems if they are operating above their threshold.

The relations between R_i and R_{out} in band-dividing FM systems are shown in figure 9. Comparing this figure with figure 8, it is clear that the band-dividing demodulator requires less intrinsic SNR than the conventional one in order to obtain an equal value of R_{out}. This comparison apparently shows that the conventional demodulator performs better than the band-dividing one at low output SNR, but as is discussed below, this results from the difference in the definition of output SNR. In both (18) and figure 9 the output noise power under modulation is used, while figure 8 is based on noise data under <u>no</u> modulation. If the output noise power under no modulation is used for band-dividing demodulators, the second term in the denominator in (18) becomes smaller by a factor (2 - Np), and it is shown that the envelope of the curves in figure 9 moves left a little such that it coincides with the envelope of the curves in figure 8 at low output SNR [Akima, 1963].

Figure 9 also shows that, in band-dividing FM systems, the value of the intrinsic SNR at the threshold is not constant but increases as the modulation index of the system increases.

Next we shall discuss the characteristics of FM systems having a frequency-lock or phase-lock demodulators. Since a mathematical analysis of these demodulators when they are operating below the threshold is a difficult problem [Enloe, 1962] we can only give the following comments.

In common in these feedback demodulators, the equivalent noise bandwidth can at best be reduced to twice the maximum frequency of the information signal f_m as in the band-dividing one. It is, therefore, only necessary to compare the mechanism of loss-of-lock in

SIGNAL-TO-NOISE CHARACTERISTICS OF
FM SYSTEMS WITH BAND-DIVIDING DEMODULATORS

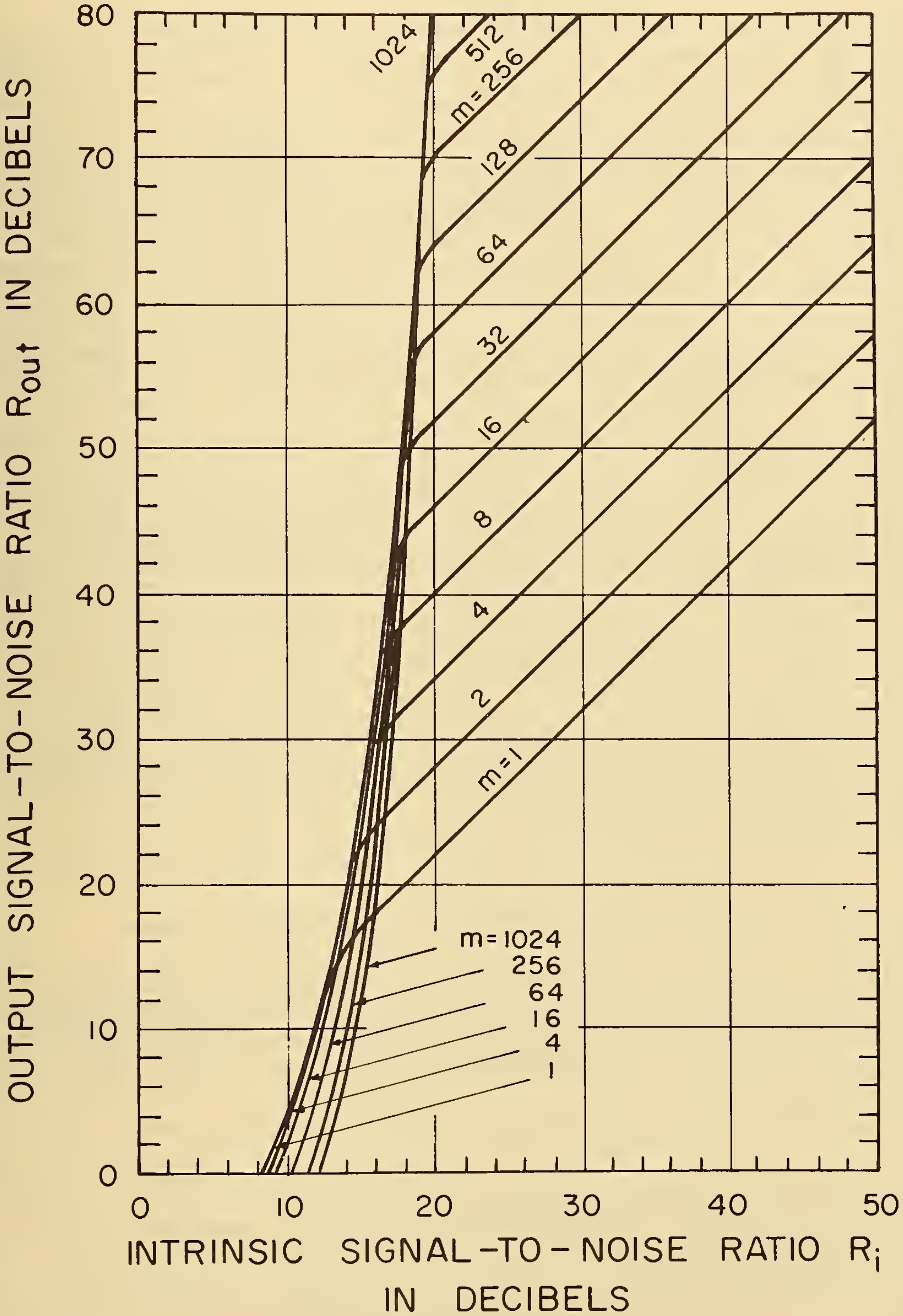


Figure 9

the feedback demodulators with the mechanism of mis-selection of the signal channel in the band-dividing demodulator. From the above descriptions the mechanism of mis-selection of the signal channel in the band-dividing one is obvious. If there is a noise channel whose band-pass filter output is larger than that of the signal channel, a mis-selection of the signal channel will take place.

Now we take into account a noise voltage in a period which is much longer than the Nyquist interval. The noise voltage can be analyzed into Fourier components, spacings of which are much smaller than f_m. If the phases of almost all the components in a specified band of width $2f_m$ coincide approximately with each other at some instant, the amplitude of the composite wave of these components is much larger than the average value, and the frequency of the composite wave is nearly equal to the center frequency of the band and does not change rapidly around that instant. As the increase of the amplitude of the composite wave cannot take place unless the phases of these components coincide with each other, we can see that the frequency of the composite wave does not change rapidly whenever the amplitude is much larger than the average value. It should be noticed that the mis-selection of the signal channel in the band-dividing demodulator takes place under these conditions. Since there is little difference in the waveform between the composite wave in the noise channel and that in the signal channel when the amplitude of the former is larger than that of the latter, it seems reasonable to consider that the feedback loop will be locked to the noise channel in such a situation. As no instantaneous phase information of the incoming signal itself is available at any demodulator in FM systems, we cannot expect a locked-in condition with less probability of loss-of-lock than the element error rate in incoherent multiple FSK systems, and therefore, we cannot expect such a condition with less probability of loss-of-lock than the probability of mis-selection of the signal channel in the band-dividing demodulator. We might, therefore, suggest that the threshold of a frequency-lock or phase-lock demodulator cannot be improved beyond that of the band-dividing one, and that the envelope of the curves in figure 9 provides the limit of improving the threshold in FM systems.

In this paper studes are made on the assumption that the carrier is frequency-modulated by a discrete signal. It is clear that the band-dividing demodulator does not apply to a continuous FM system but only to a discrete one. We might, however, suggest that the threshold of a continuous FM system cannot be improved beyond that of a discrete system given in figure 9, because both systems can be considered to yield exactly the same information, by Shannon's sampling theorem [Shannon, 1948], and no instantaneous phase information of the signal itself is available in either case.

4. SIGNAL-TO-NOISE CHARACTERISTICS OF PCM-FS SYSTEMS

In this section we shall discuss the signal-to-noise characteristics of PCM-FS systems. In these systems an analog information signal is sampled at every sampling point, equally spaced by a Nyquist interval, and the sampled value is quantized into L equally spaced levels, and the quantized signal is coded into n elements with the base in the coding N. The relations between L, N, and n is given by

$$L = N^n. \tag{20}$$

The coded signal is used to frequency-modulate the carrier. It may be sent over one N-ary FSK link in time sequence or over n N-ary FSK links by frequency division [Oliver, et al., 1948]. As the modulating signal is quantized, either the coherent or incoherent FSK system can be used. At the receiver the incoming wave is demodulated by an FSK demodulator or demodulators, and the information signal is recovered by decoding the demodulated signal.

As the Nyquist interval corresponding to the maximum frequency of the information signal f_m is equal to $1/(2f_m)$ and n elements are transmitted in each interval, the bandwidth of each channel in the FSK demodulator B_c is given by

$$B_c = 2nf_m = 2nB_i \tag{21}$$

and

$$B_c = 2f_m = 2B_i, \tag{22}$$

corresponding to the transmissions in time sequence and by frequency division, respectively. The overall bandwidth of the system B_a is given by

$$B_a = 2nNf_m = 2nNB_i, \tag{23}$$

and the relation between the channel SNR R_c and the intrinsic SNR R_i is given by

$$R_i = 2nR_c = 2(\log_N L)R_c, \tag{24}$$

irrespective of the two schemes of transmission.

If errors in the FSK transmission (or mis-selections of the signal channel) take place, an additional output noise power will be produced besides the quantizing noise, and the phenomenon of "modulation suppression" will also take place. In Appendix C it is shown that the output SNR R_{out} is given by

$$R_{out} = \frac{3}{2}(L-1)^2 \frac{(1 - Np)^2}{1 + (L^2-1)(2 - Np)Np}, \tag{25}$$

where p is the probability that any noise channel is selected as the signal channel by mistake, and it is related to the element error rate p_e in an N-ary FSK system by

$$p = p_e/(N - 1), \tag{26}$$

as in the band-dividing FM system. As P_e is given as a function of the channel SNR R_c and the number of frequencies N, we can determine the relation between the intrinsic SNR R_i and the output SNR R_{out} from the relations given above.

When the input SNR is so large that the probability of mis-selection p is negligible, the output SNR R_{out} is given by

$$R_{out} = \frac{3}{2}(L - 1)^2. \tag{27}$$

This relation shows that the maximum output SNR in this system depends only on the number of quantizing levels L and not on the base in the coding N.

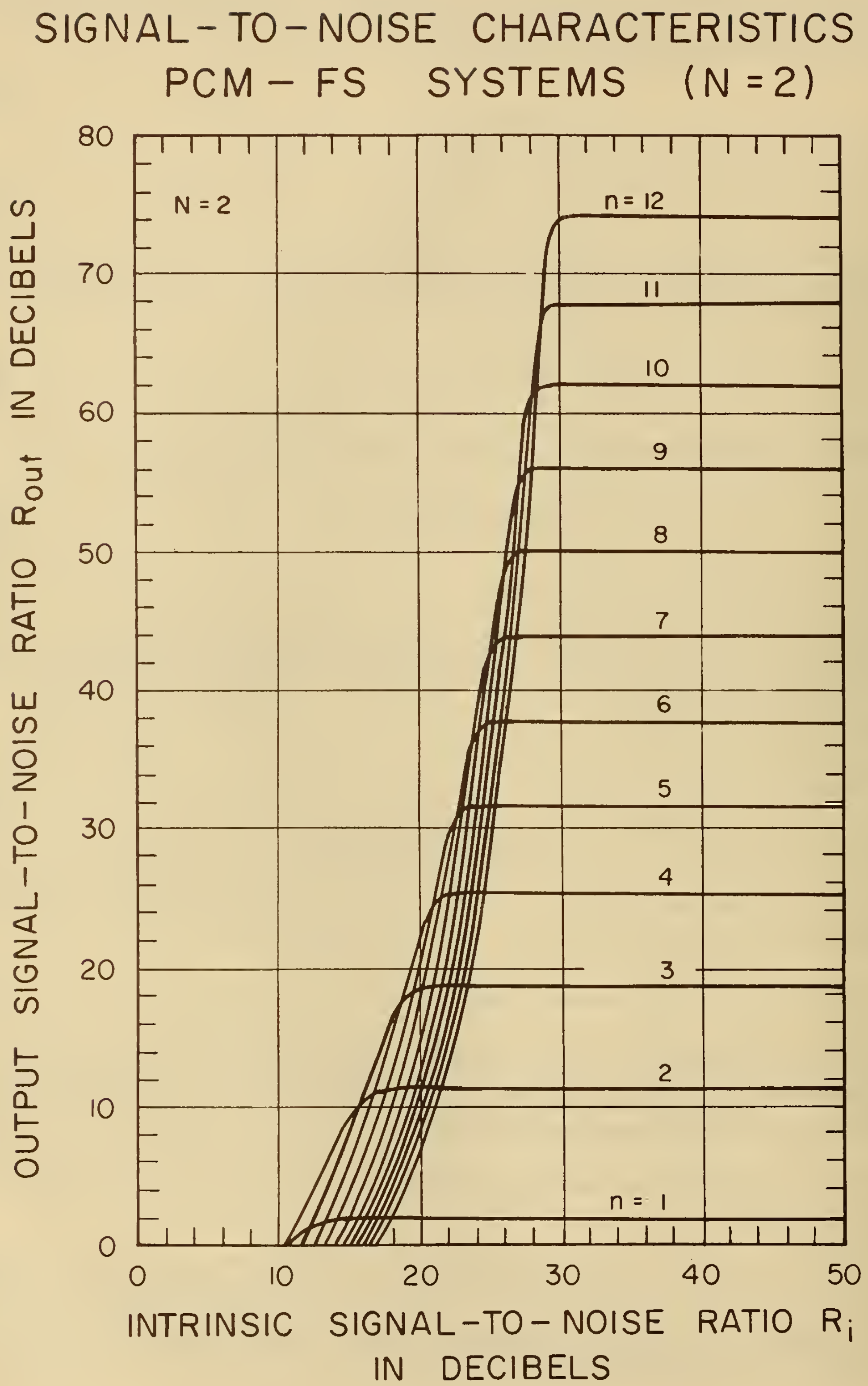

SIGNAL-TO-NOISE CHARACTERISTICS OF PCM-FS SYSTEMS (N=2)
N = 2
n = 12
11
10
9
8
7
6
5
4
3
2
n = 1
0
10
20
30
40
50
60
70
80
OUTPUT SIGNAL-TO-NOISE RATIO R_{out} IN DECIBELS
INTRINSIC SIGNAL-TO-NOISE RATIO R_i IN DECIBELS

Figure 10

SIGNAL - TO - NOISE CHARACTERISTICS OF PCM - FS SYSTEMS (N = 3)

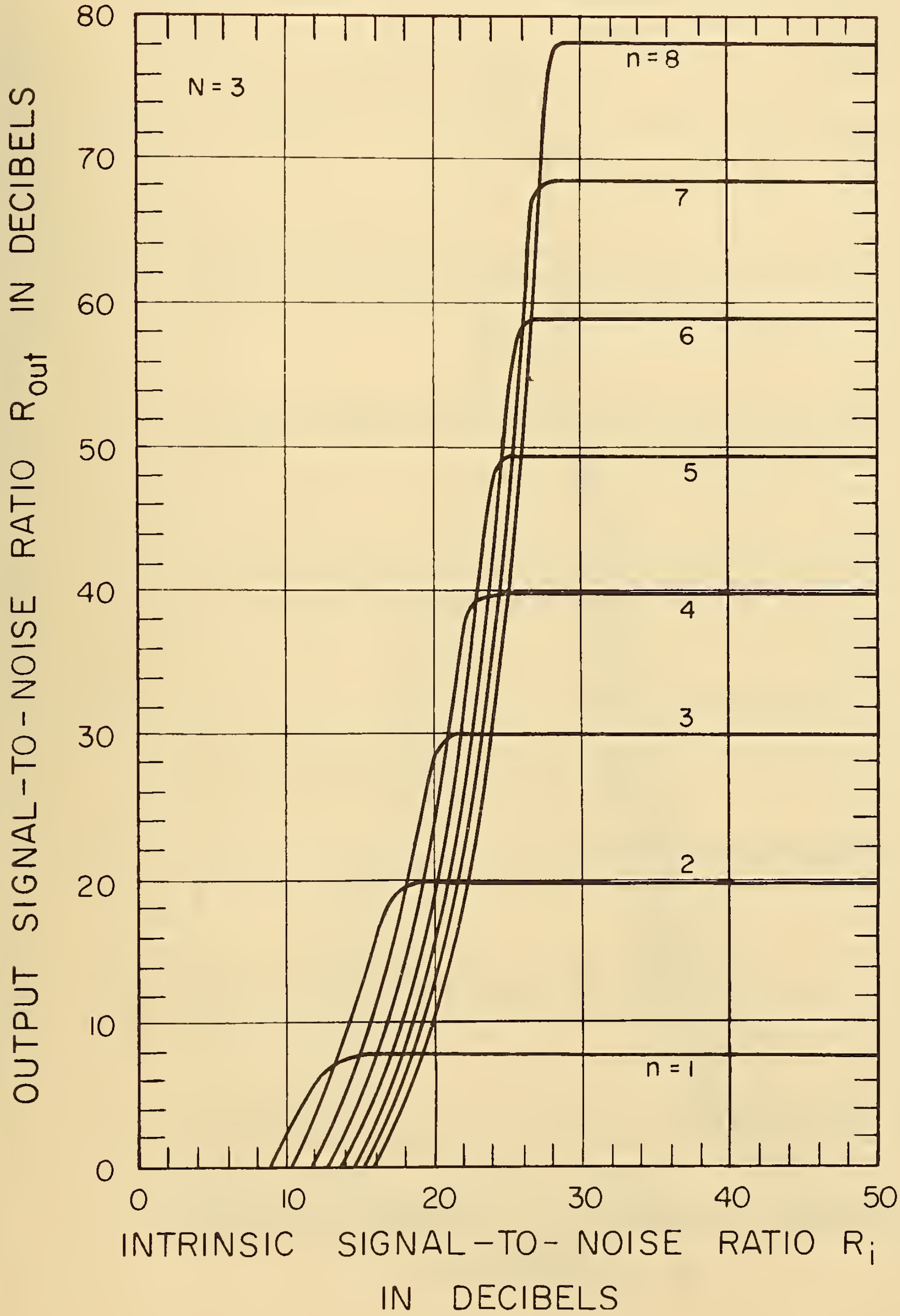


Figure 11

SIGNAL-TO-NOISE CHARACTERISTICS OF PCM - FS SYSTEMS (N = 4)

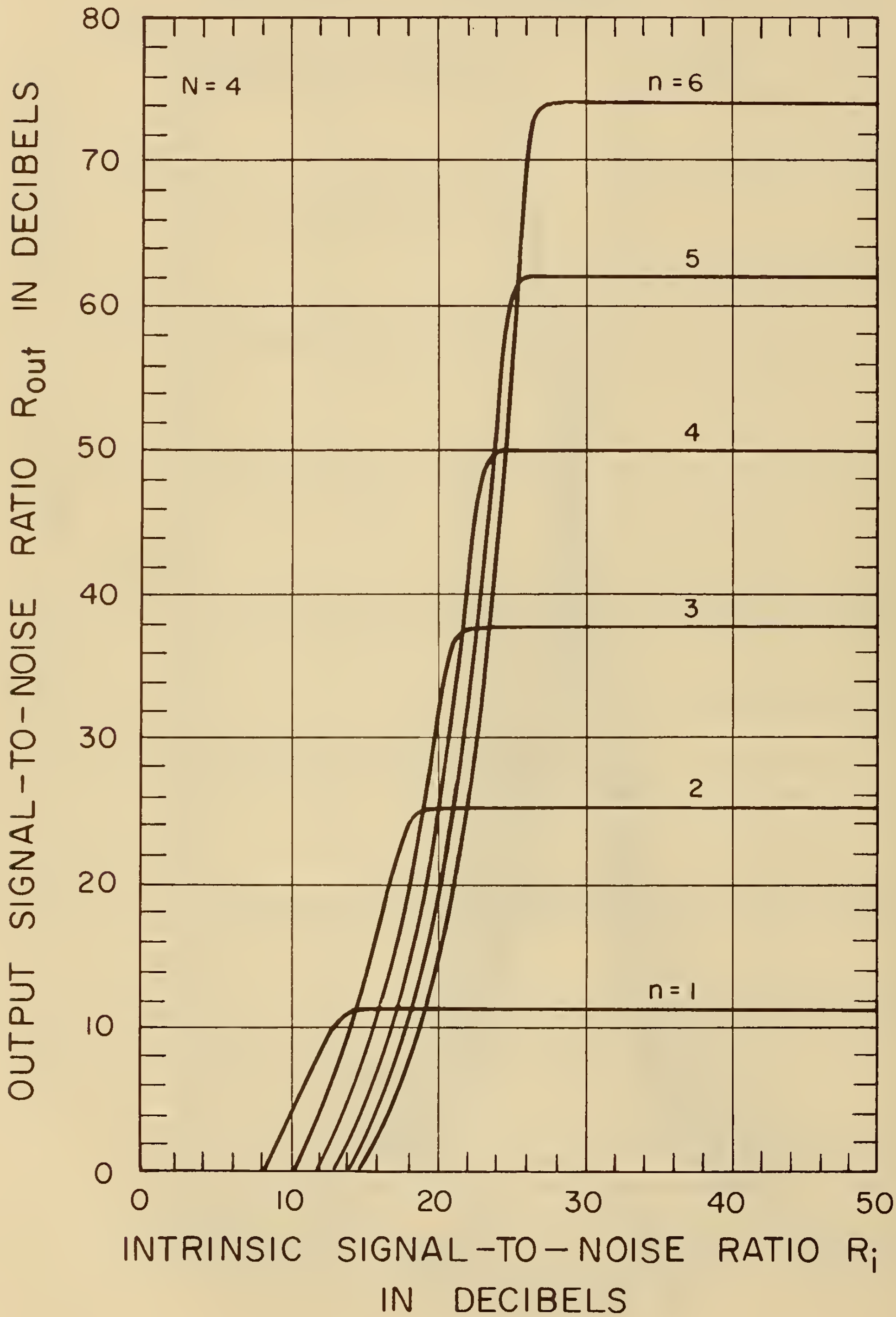


Figure 12

SIGNAL-TO-NOISE CHARACTERISTICS OF PCM-FS SYSTEMS (N = 8)

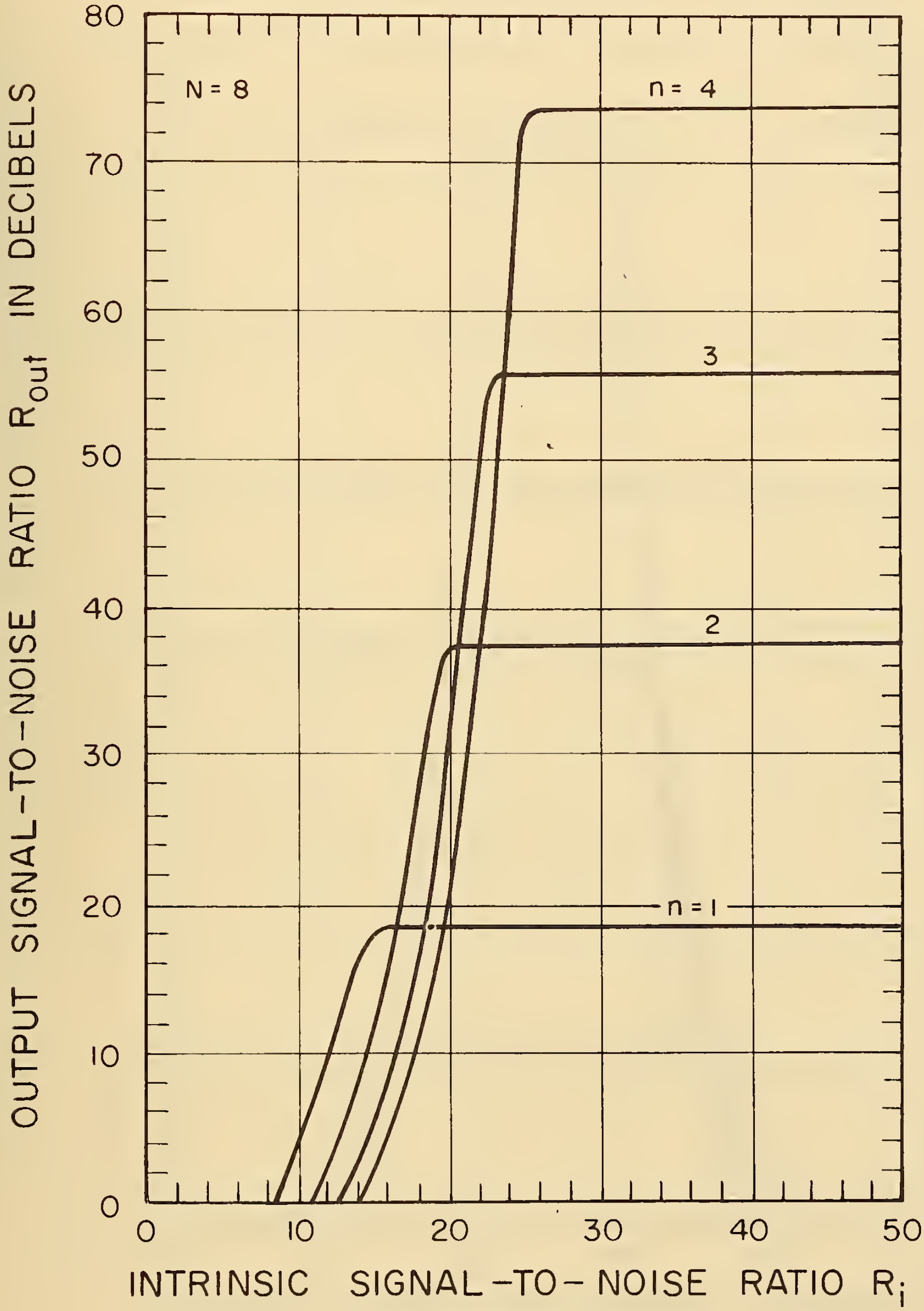


Figure 13

SIGNAL-TO-NOISE CHARACTERISTICS OF PCM — FS SYSTEMS (n = 1)

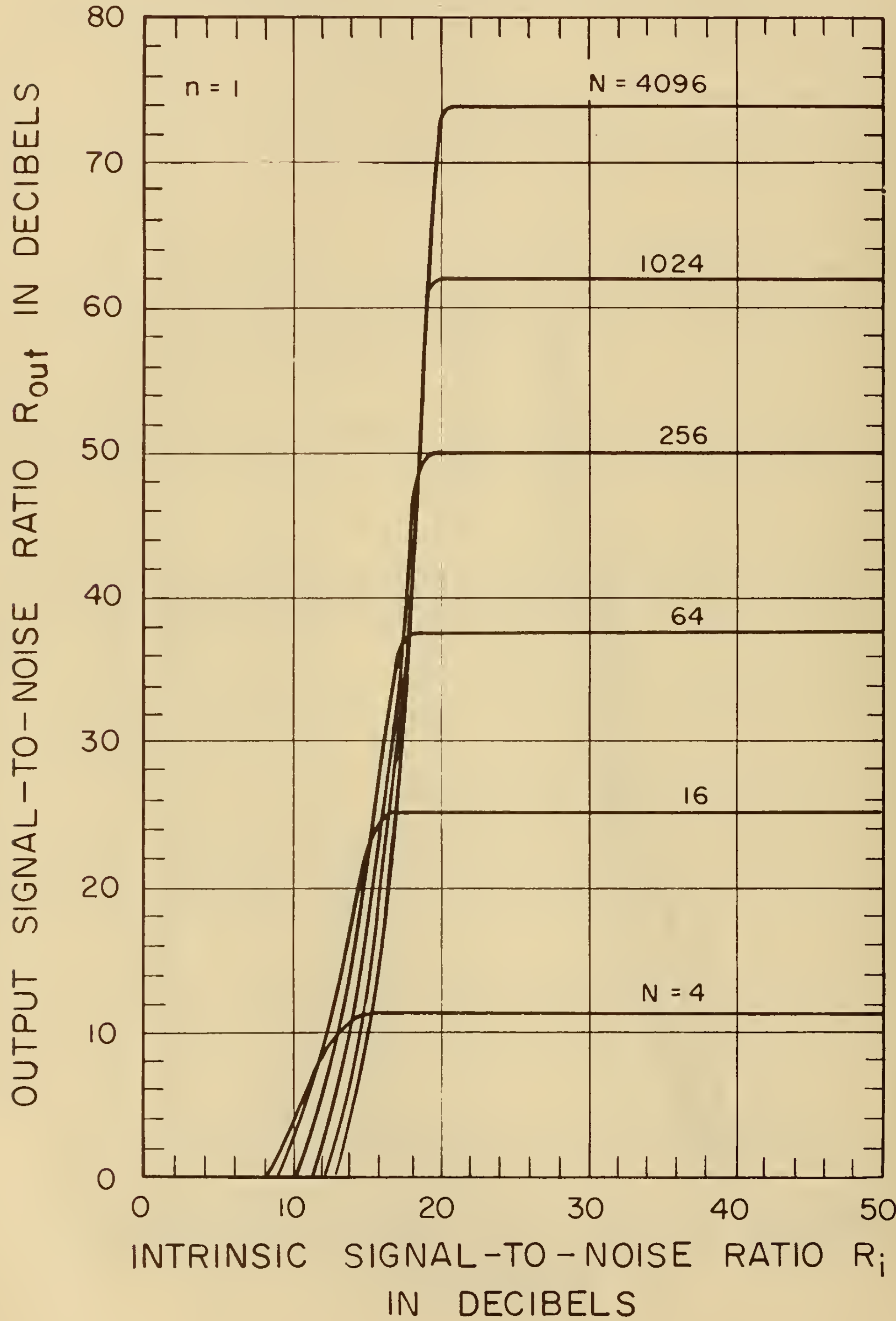


Figure 14

SIGNAL-TO-NOISE CHARACTERISTICS OF PCM-FS SYSTEMS (n = 2)

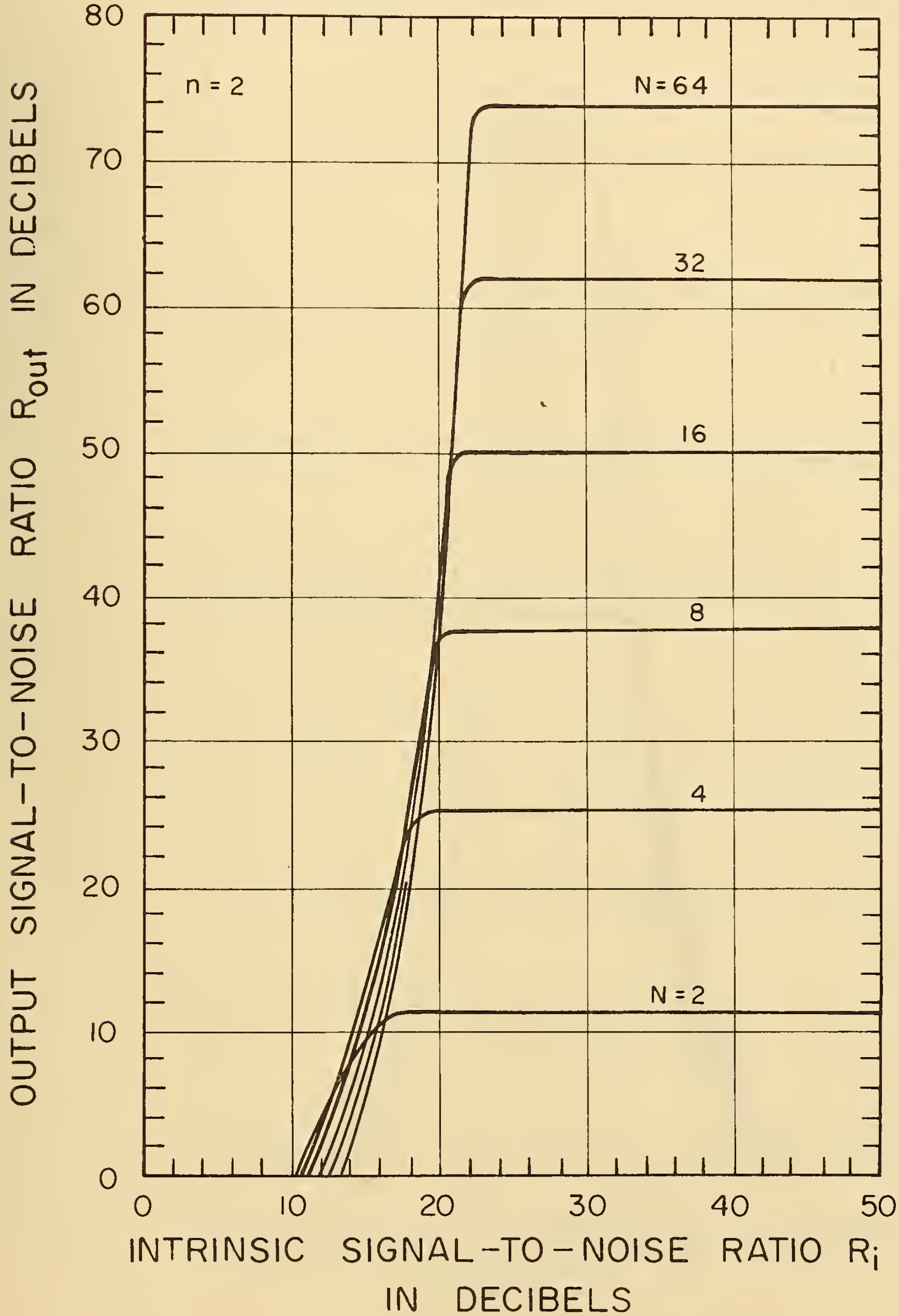


Figure 15

SIGNAL - TO - NOISE CHARACTERISTICS OF PCM - FS SYSTEMS (n = 3)

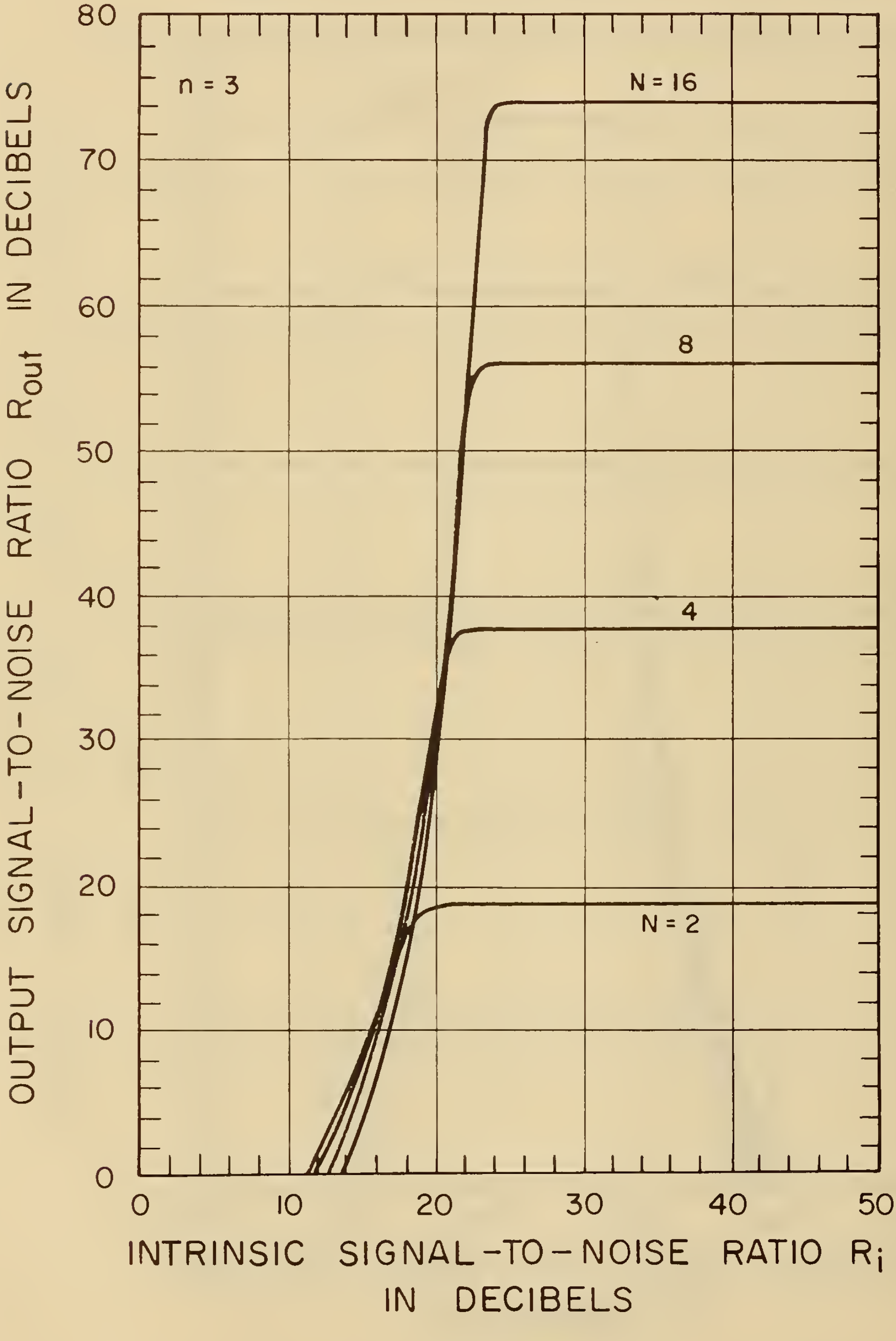


Figure 16

SIGNAL-TO-NOISE CHARACTERISTICS OF PCM-FS SYSTEMS (n = 4)

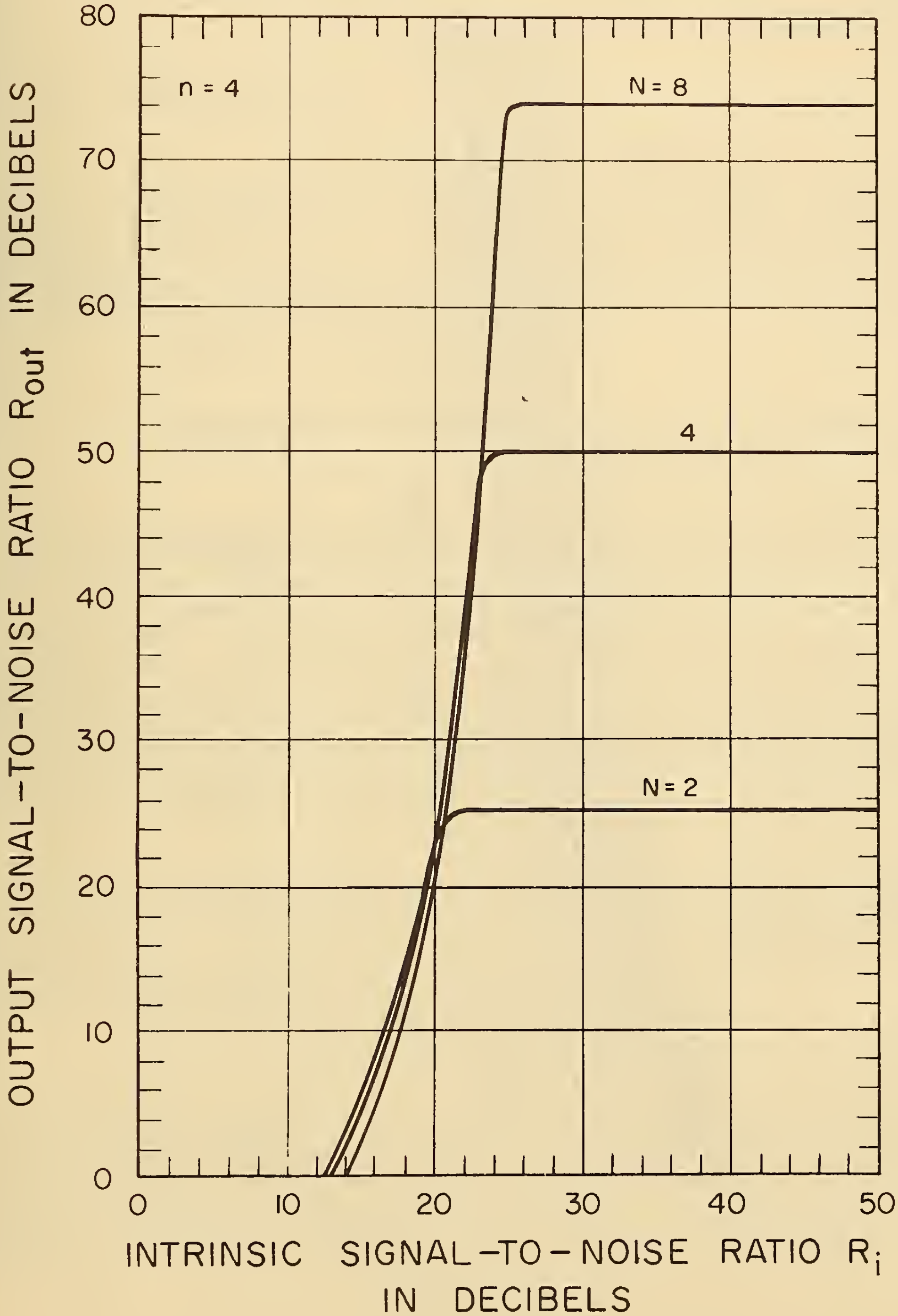


Figure 17

SUMMARIZED THRESHOLD CURVES
OF PCM-FS SYSTEMS

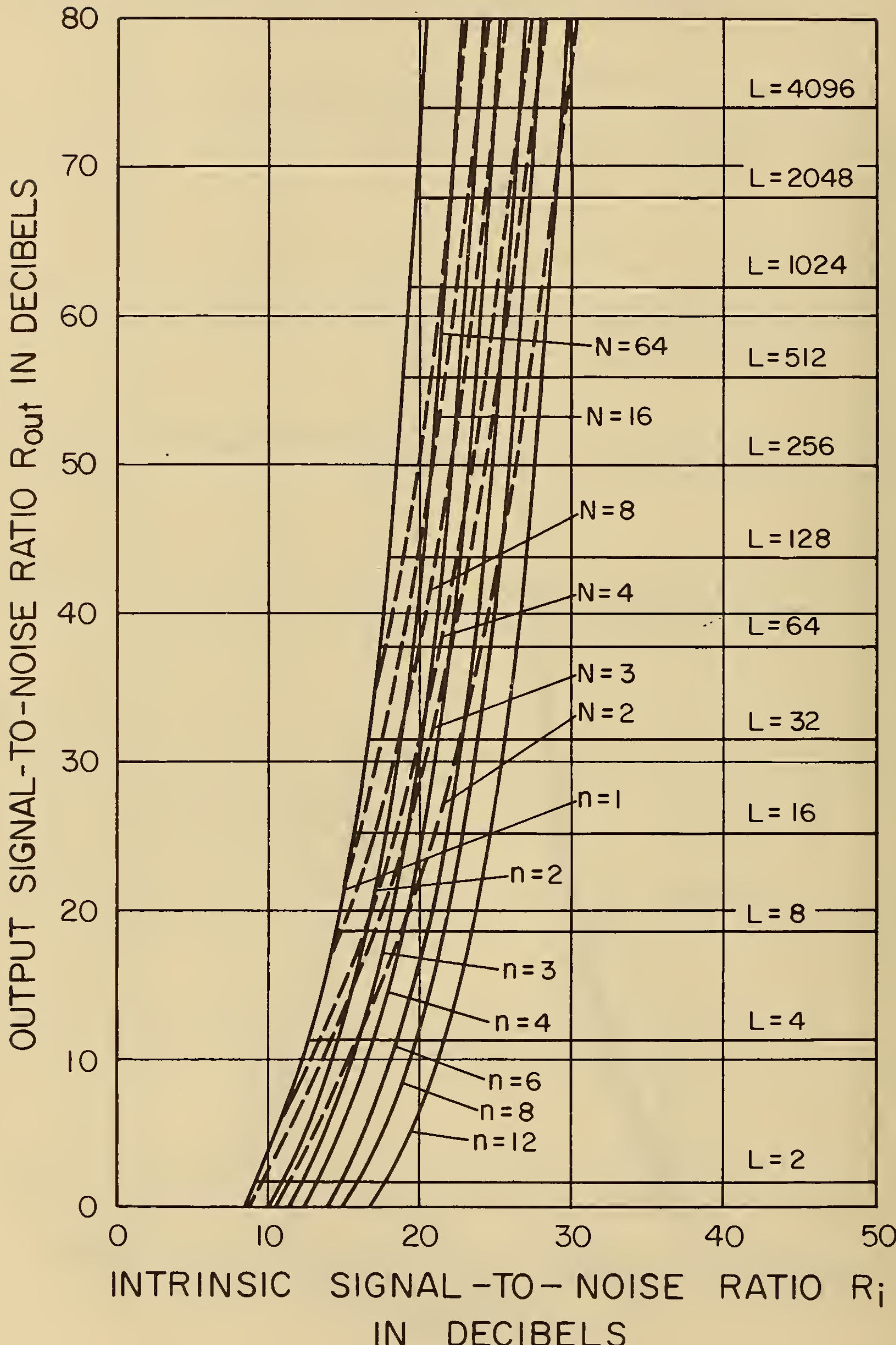


Figure 18

The signal-to-noise characteristics of PCM-FS systems with incoherent detectors are calculated and shown in figures 10 - 17. Figures 10, 11, 12, and 13 correspond to the values of the base N equal to 2, 3, 4, and 8, respectively, whereas figures 14, 15, 16, and 17 correspond to the number of elements n equal to 1, 2, 3, and 4, respectively. An envelope curve can be drawn in each of these figures, and these envelope curves are summarized in figure 18. This figure can be considered to provide the lower limit to the required intrinsic SNR (or the required signal power) in PCM-FS systems. It is clear from the figure that the required intrinsic SNR can be reduced by increasing the base N. This conclusion is exactly in parallel relation with the one in multiple FSK systems.

From (20) and (23) the relation

$$B_a/B_i = (2 \log_2 L)(N/\log_2 N) \tag{28}$$

is derived. Comparing (28) with (9) in Section 2 we can notice that the comment on the band occupancy in multiple FSK systems can also apply to PCM-FS systems. It must, therefore, be stressed that a ternary or quaternary PCM-FS system can achieve the reduction of required intrinsic SNR beyond a binary one without requiring a wider overall bandwidth B_a for a given number of quantizing levels L.

5. COMPARISONS BETWEEN SSB, FM, AND PCM-FS SYSTEMS

5.1. General Considerations

Next we shall compare the signal-to-noise characteristics of SSB, FM, and PCM-FS systems from the standpoint of communication system engineering, because these systems are the most typical systems in analog information transmission.

Consider the situation in which the same information signal is transmitted over each channel between the same two points in each system such that the same quality, i.e., the same output SNR in the analog case can be obtained at the common destination. We then determine the overall bandwidth and the intrinsic SNR (or the signal power) required in each channel, calculate the cost of each channel, and compare the cost of channels with each other. A system which requires a less expensive channel to achieve an equal output SNR is regarded as the better system, insofar as the signal-to-noise characteristics are concerned.

If both the required bandwidth and the required intrinsic SNR (or signal power) in the channel in one system are smaller than those in another system, the cost of the channel in the former system is lower than that in the latter, and therefore, the former system can be regarded as the absolutely better system. It is, however, not the case in general. As a rule we must make conditional comparisons by the cost of the required channel in each system. In the following we shall make comparisons with some simple criteria.

5.2. Absolute Comparisons

As mentioned before there exists no absolutely best system across the whole range of the output SNR even if our attention is confined to the signal-to-noise characteristics. In designing a communication system, however, the minimum required output SNR is assigned at the beginning, and the absolute comparison can sometimes be made for an assigned value of the output SNR. In practice, for example, we would not use a PCM system when a relatively low output SNR, say 30 db or below, is required, because we have better systems than PCM in such a situation.

To make an absolute comparison for an assigned value of the output SNR R_{out}, it is

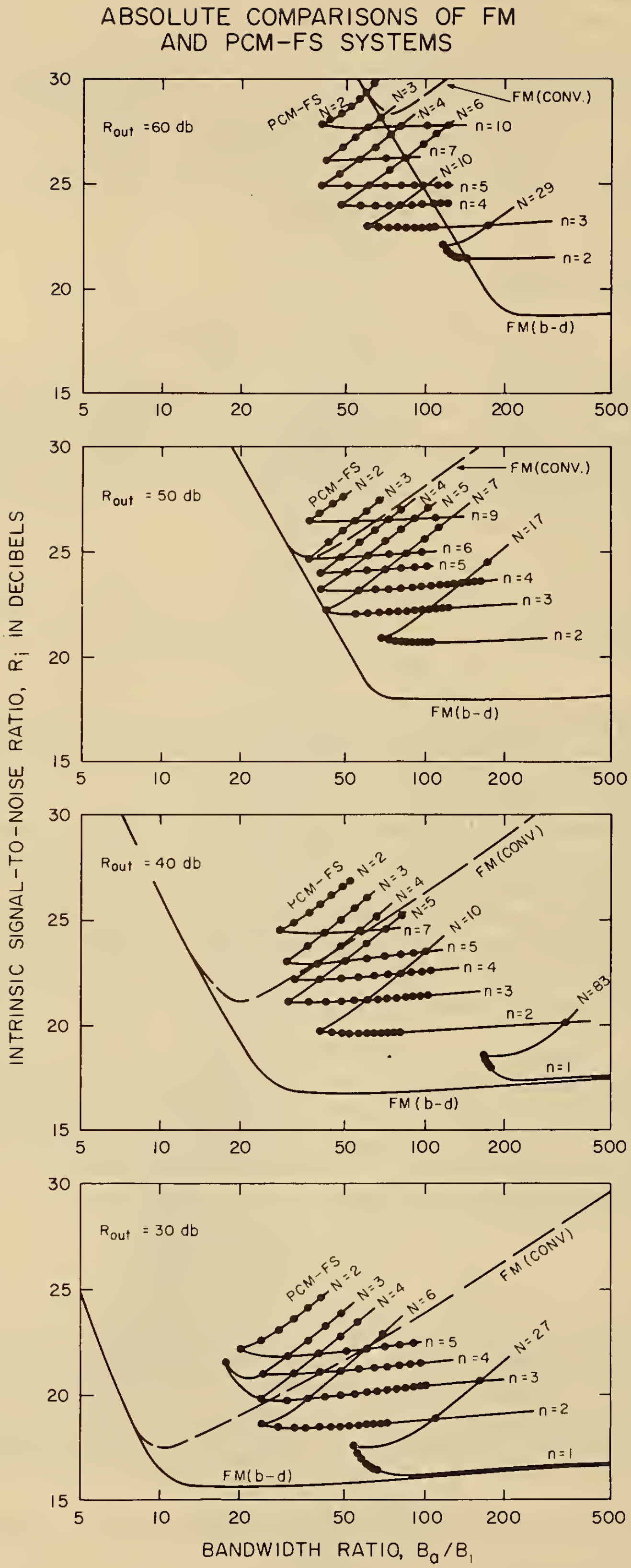

ABSOLUTE COMPARISONS OF FM AND PCM-FS SYSTEMS
INTRINSIC SIGNAL-TO-NOISE RATIO, R_i IN DECIBELS
R_{out} =60 db
PCM-FS
N=2
N=3
N=4
N=6
N=10
N=29
FM(CONV.)
n=10
n=7
n=5
n=4
n=3
n=2
FM(b-d)
R_{out} = 50 db
N=5
N=7
N=17
n=9
n=6
R_{out} = 40 db
N=83
n=1
R_{out} = 30 db
N=27
BANDWIDTH RATIO, B_a/B_i

Figure 19

convenient to replot the signal-to-noise characteristics in a B_a/B_i - R_i plane, where B_a and B_i are the overall bandwidth of the channel and the intrinsic bandwidth of the information signal, respectively, and R_i is the intrinsic SNR. If we can observe from the figure that both B_a/B_i and R_i in one system are smaller than those in another, we can conclude that the former system has an absolute superiority over the latter for the assigned value of R_{out}.

In figure 19 the signal-to-noise characteristics of FM and PCM-FS systems are replotted in B_a/B_i - R_i planes for the values of R_{out} equal to 30, 40, 50, and 60 decibels. For R_{out} = 30 db, even if a conventional FM demodulator is used, an FM system is better than a PCM-FS system with the number of elements for each sample n equal to or greater than 2. For R_{out} = 40 db, however, a PCM-FS system with n = 2 or 3 can be competitive with a conventional FM system. For R_{out} = 50 db, a PCM-FS system with n up to 6 can be competitive with a conventional FM system, but a band-dividing FM system can still be absolutely better than PCM-FS systems. For R_{out} = 60 db, the situation is completely different from the above, and a PCM-FS system can be competitive with a band-dividing FM system. It must be noticed that the absolute superiority of a PCM-FS system to a band-dividing FM system can never be observed for any value of the output SNR R_{out}.

5.3. Comparisons with the Minimum-Bandwidth Criterion

Sometimes the requirement for narrow bandwidth is so definite that the minimum-bandwidth criterion can apply. In this case it is convenient to replot the signal-to noise characteristics in a B_a/B_i - R_{out} plane, and to compare the required bandwidth ratio B_a/B_i to obtain an equal value of the output SNR R_{out} under an equal value of the intrinsic SNR R_i.

In figure 20 the signal-to-noise characteristics of SSB, FM, and PCM-FS systems are replotted in B_a/B_i - R_{out} planes under the values of R_i equal to 20, 25, and 30 decibels. Although an SSB system requires the narrowest bandwidth among the systems compared here, it does not achieve any broadband gain, i.e., the same value of R_i as that of R_{out} is required in this system. Our main interest is, therefore, in the comparison between FM and PCM-FS systems.

When R_i is equal to or larger than 30 db, both the FM and PCM-FS systems operate in the above-the-threshold region, and a PCM-FS system can be better than a band-dividing FM system for relatively large values of R_{out}. The comparison between PCM and FM with the use of broadband gain by Oliver, et. al. [1948] roughly corresponds to the comparison given here. As stated in their paper it is important to note that, as the bandwidth ratio B_a/B_i in our notation is increased, R_{out} expressed in decibels varies as $\log(B_a/B_i)$ in an FM system, while it varies as B_a/B_i in a PCM-FS system.

When R_i is equal to or smaller than 20 db, on the other hand, the threshold in a PCM-FS system takes place before a large amount of broadband gain is obtained, and a PCM-FS system requires a wider bandwidth than a band-dividing FM system.

COMPARISONS OF SSB, FM, AND PCM-FS SYSTEMS WITH THE MINIMUM-BANDWIDTH CRITERION

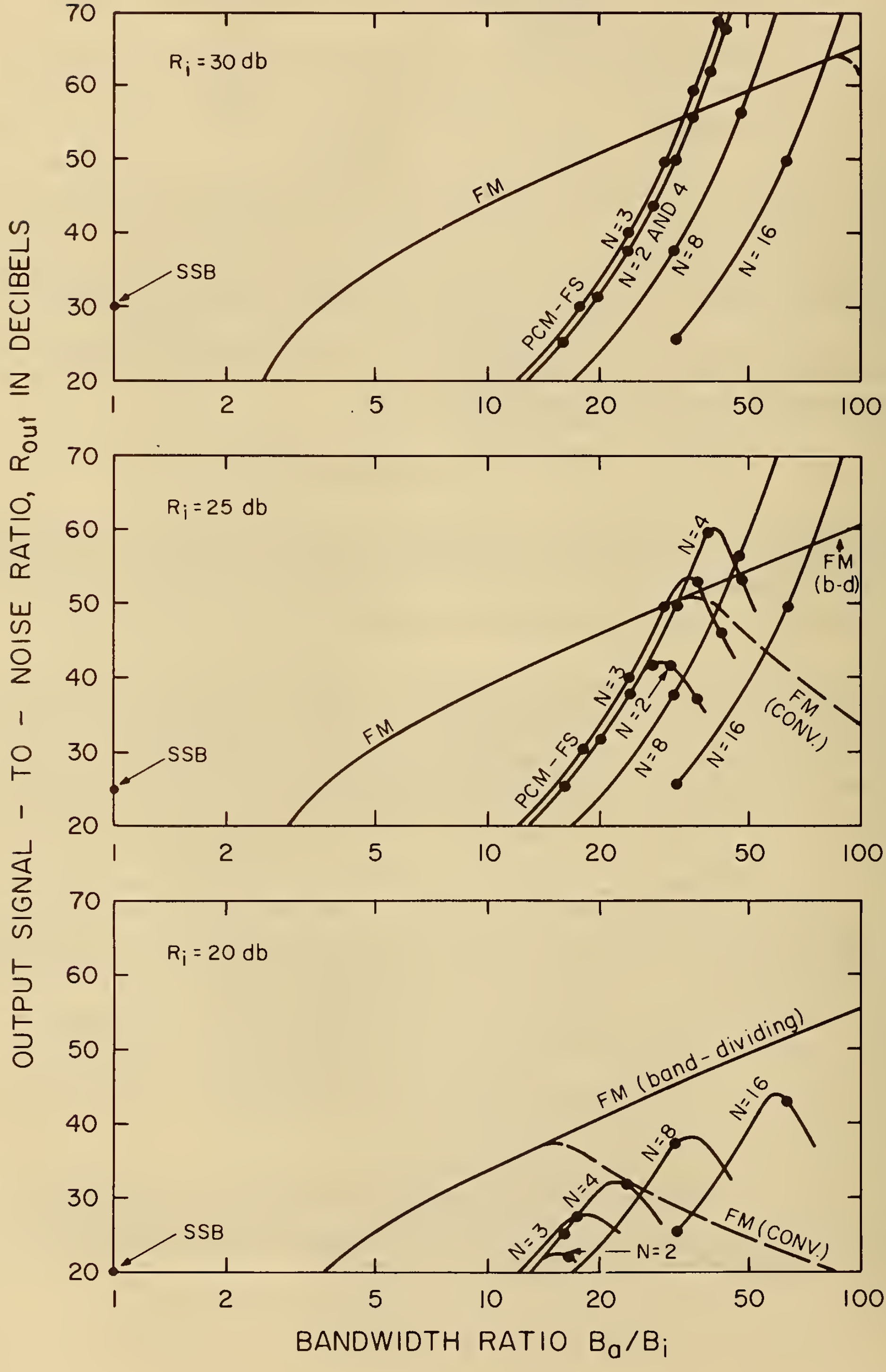


Figure 20

5.4. Comparisons with the Minimum-Power Criterion

On the contrary to the preceding comparison the cost of the channel is sometimes determined essentially by the signal power. A typical case is a communication from a satellite, where transmitter power is severely limited, while the requirement for narrow bandwidth is not so severe. To make system comparisons with the minimum-power criterion the representation of the signal-to-noise characteristics in an R_i - R_{out} plane is very convenient, because the intrinsic SNR R_i is equivalent to the signal power. This representation has been used throughout sections 3 and 4 of this paper. In figure 18 PCM-FS systems having different values of system parameters were compared with each other. It is an example of comparisons with minimum-power criterion.

In each system the minimum power for an assigned value of R_{out} is obtained when the system is operating on the threshold curve, i.e., on the envelope of the curves in figures 8 to 17. In order to make system comparisons with the minimum-power criterion it is sufficient to compare only these threshold curves. In figure 21 the threshold curves of FM systems are compared with some of those of PCM-FS systems. In this figure the characteristic of an SSB system is also shown as a reference. As mentioned in section 3, it should be noted that the curve for the conventional FM system drawn with a broken line is based on noise data under <u>no</u> modulation, whereas other curves are based on those under modulation. It is clear from this figure that an FM system having a band-dividing demodulator is the best system among the systems compared here insofar as the minimum-power criterion is concerned at output SNR greater than about 10 decibels.

From (17) and (18) the output SNR R_{out} in a band-dividing FM system can also be written as

$$R_{out} = \frac{3}{2}(N-1)^2 \frac{(1-Np)^2}{1/R_i + (N^2-1)(2-Np)Np} \tag{29}$$

In PCM-FS systems, on the other hand, the required intrinsic SNR can be reduced by increasing the base in the coding N, as clear from figure 18. In the limit where $N = L$ and $n = 1$, equation (25) for the output SNR R_{out} in the PCM-FS system can be modified as

$$R_{out} = \frac{3}{2}(N-1)^2 \frac{(1-Np)^2}{1 + (N^2-1)(2-Np)Np} \tag{30}$$

Comparing (29) with (30) we can see the difference, as well as the similarity, between these systems. It is noticed that the difference between these equations is only in the first term in the denominator and the other terms are exactly the same. The first terms in the denominators in (29) and (30) correspond to the output noise due to incoming noise into the signal channel in a band-dividing FM system and the quantizing noise in a PCM-FS system, respectively. Although the latter term is a constant, the former term is a reciprocal of R_i and is smaller than the latter whenever R_i is greater than unity, or zero decibels. As relations between p and R_i are the same in these systems, it can be concluded that, in order to obtain an equal value of the output SNR, the required intrinsic SNR in a PCM-FS system with $N = L$ cannot be

COMPARISONS OF SSB, FM, AND PCM-FS SYSTEMS WITH THE MINIMUM-POWER CRITERION

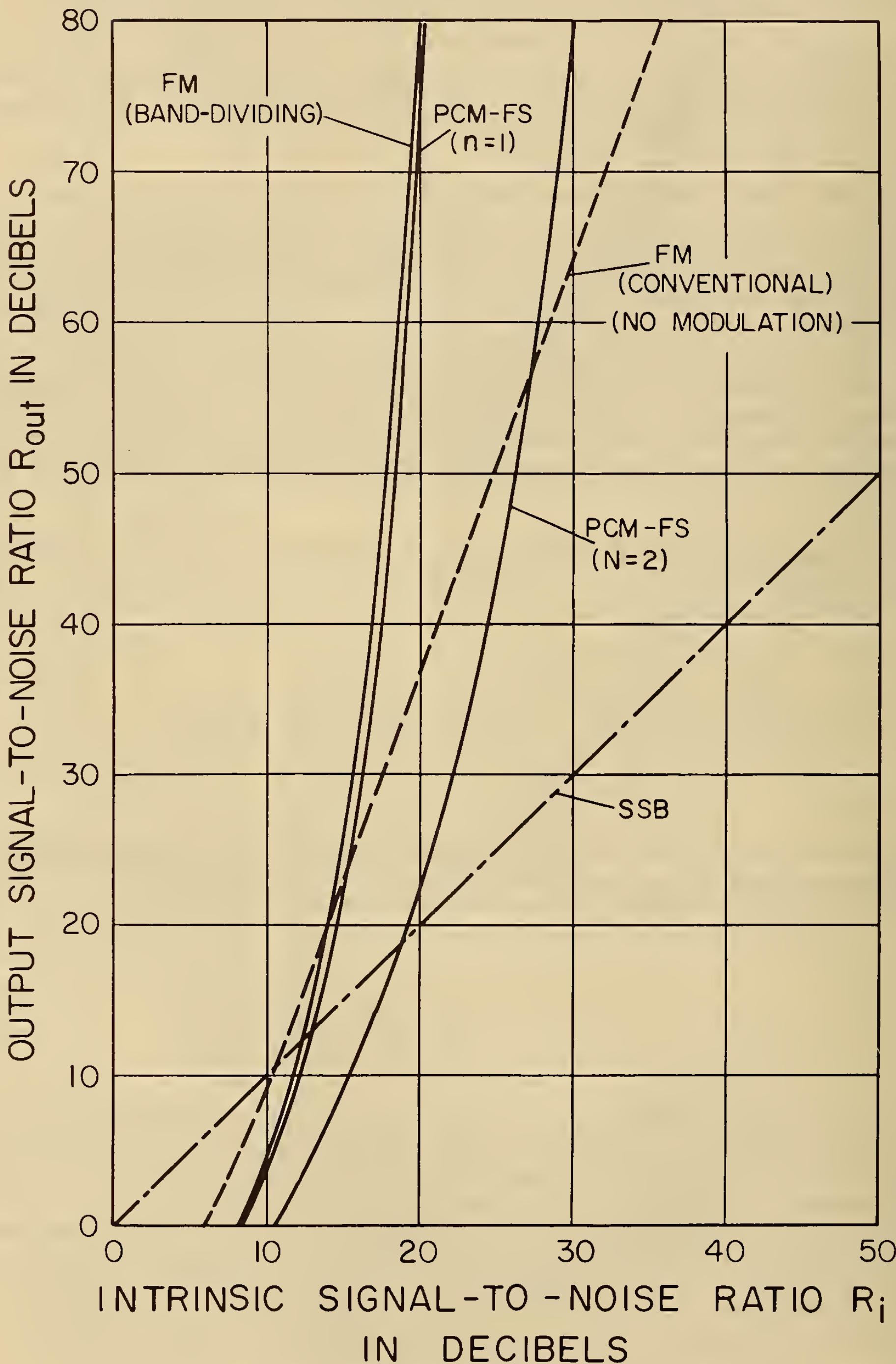


Figure 21

smaller than the required intrinsic SNR in a band-dividing FM system with the same value of N whenever the intrinsic SNR is larger than zero decibels. Thus the superiority of a band-dividing FM system to a PCM-FS system is shown theoretically, too.

It is also shown in figure 21 that, even if a conventional demodulator is used, an FM system is better than a binary PCM-FS system unless the assigned output SNR R_{out} is larger than approximately 55 decibels.

5.5. Comparisons with the Minimum-Channel-Capacity Criterion.

Next, we shall make comparisons from the standpoint of the channel capacity.

The channel-capacity theorem due to Shannon [1948] states that, if the rate of transmission of information is smaller than a certain value called the channel capacity, it is possible to send the information through the channel with an arbitrarily high reliability. The theorem also states that, inversely, it is impossible to send the information through a channel at a transmission rate greater than the channel capacity with an arbitrarily high reliability. Thus the channel capacity of a channel is the maximum rate of transmission of information through the channel, and therefore, it is one of the inherent properties of the channel.

Although there have been many discussions on the rate of transmission of communication systems [Jelonek, 1952; etc.], we shall discuss the problem in a somewhat different manner. Here we shall not consider the question, "Which system can transmit more information through a channel with a given capacity?" Instead, we shall pose another question, "In order to obtain an equal value of the output SNR, which system requires a channel with less capacity?"

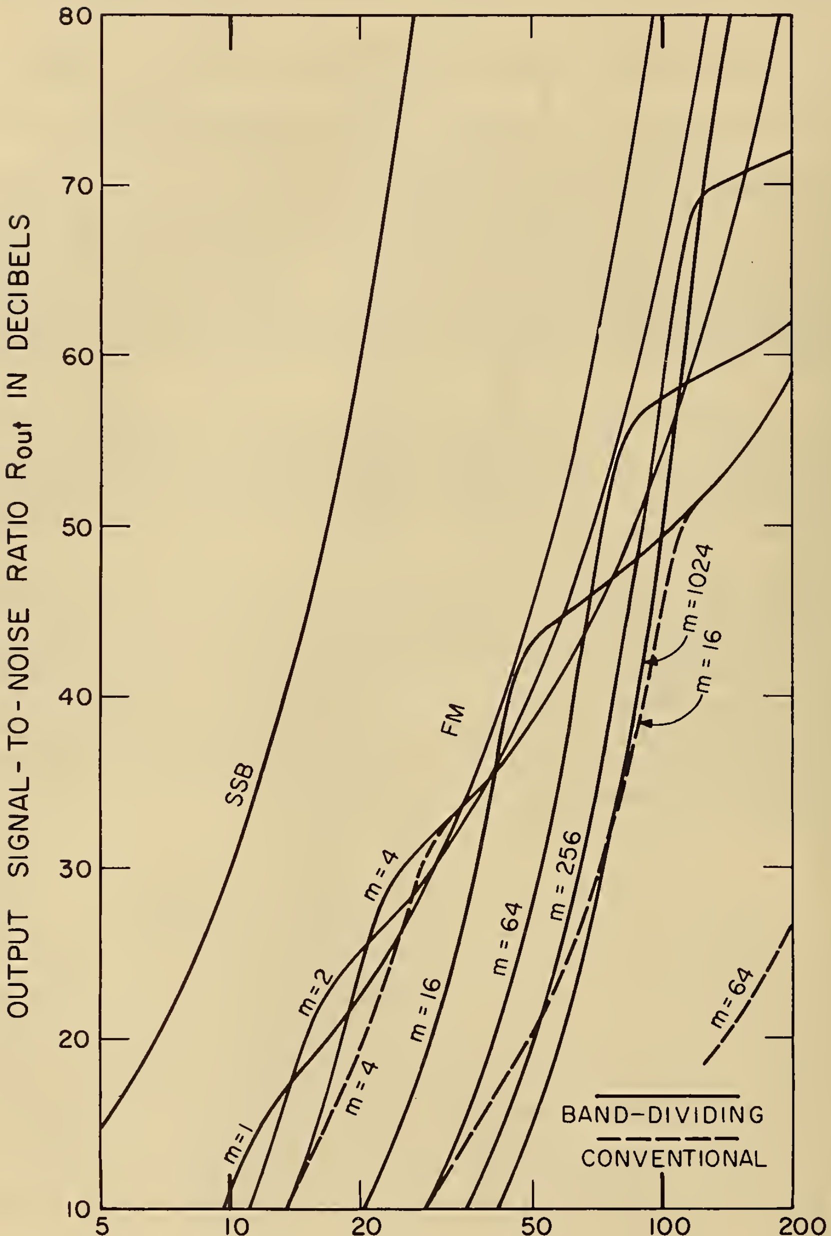
The logic behind the question is as follows: As mentioned above, the channel capacity is an inherent property of the channel like bandwidth or SNR. If a channel has the same value of capacity C as another channel, the two channels are equivalent insofar as the potential ability of transmitting the information is concerned. Therefore, if a system requires a channel with larger capacity than another system, the former system is considered to be more expensive or luxurious, and hence to be poorer theoretically. In other words we assume that the cost of the channel to be used in a system is given as a monotonic increasing function of the channel capacity. In our notation the channel capacity C is given by

$$C = B_a \log_2(1 + R_a) = B_a \log_2\left(1 + \frac{R_i}{B_a/B_i}\right), \tag{31}$$

where R_a is the overall SNR at the output of the channel or the input of the demodulator and is defined as the ratio of the signal power to the noise power contained in a band of width B_a. If we take a ratio of the required channel capacity C to the intrinsic bandwidth of the information signal B_i, the ratio C/B_i can be considered to be a measure of channel occupancy of communication systems. As the ratio C/B_i is determined by the intrinsic SNR R_i and the bandwidth ratio B_a/B_i, we can calculate this ratio for an assigned value of the output SNR R_{out} in various systems.

In figures 22 and 23 the required channel capacities in FM and PCM-FS systems are compared with that in an SSB system, respectively. It is clear from these figures that band-dividing FM systems are better than PCM-FS, and that an SSB system is the best of all, insofar as the minimum-channel-capacity criterion is concerned. It should, therefore, be

COMPARISONS OF THE SSB AND FM SYSTEMS WITH THE MINIMUM-CHANNEL-CAPACITY CRITERION.



RATIO OF THE REQUIRED CHANNEL CAPACITY TO THE INTRINSIC BANDWIDTH OF THE INFORMATION SIGNAL C/B_i

Figure 22

COMPARISONS OF THE SSB AND PCM-FS SYSTEMS WITH THE MINIMUM-CHANNEL-CAPACITY CRITERION.

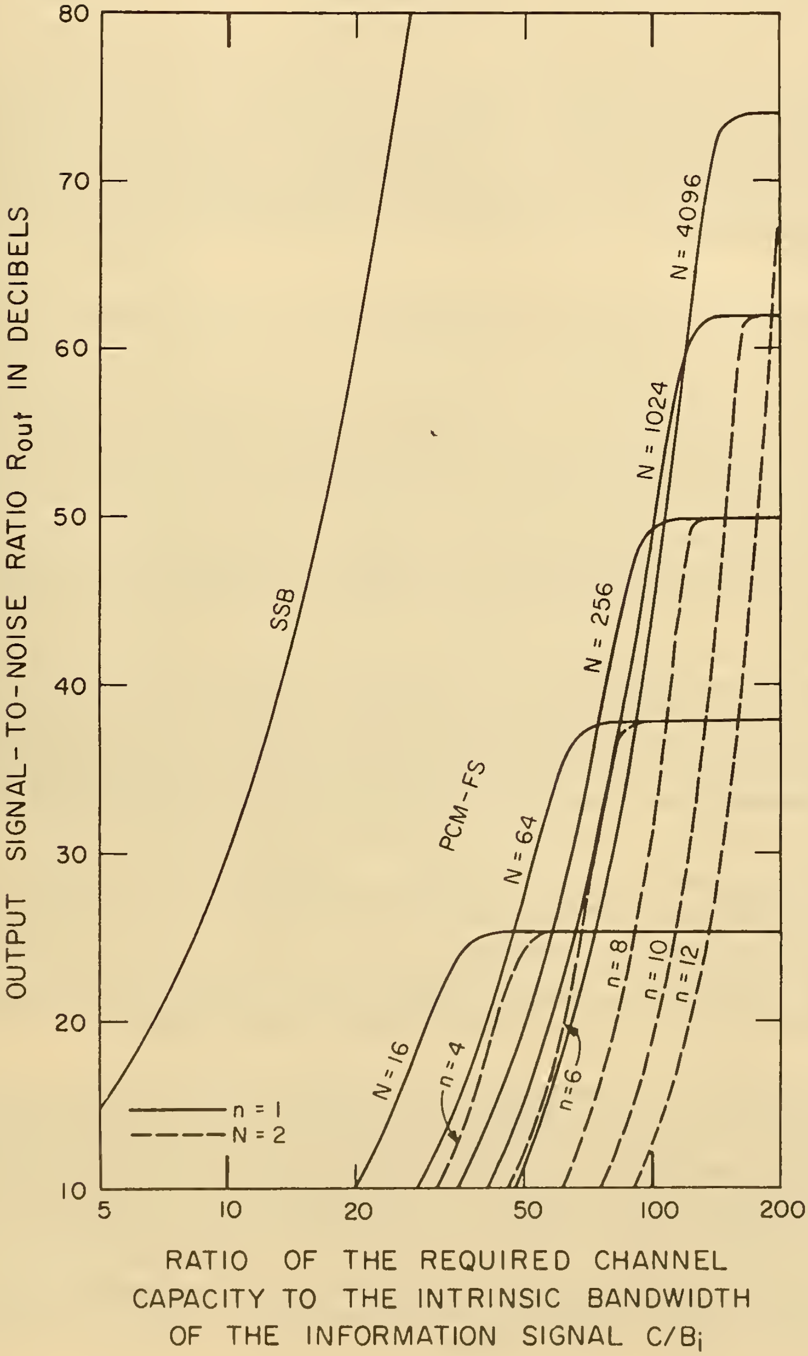


Figure 23

recognized that the so-called broadband systems like FM or PCM do not improve the efficiency in the channel capacity but only achieve the interchange or trade between bandwidth and signal power to some extent.

6. CONCLUSIONS

The element and symbol (or character) error rates in multiple FSK systems are evaluated theoretically, and based on the error studies in multiple FSK systems, the studies are made on the essential signal-to-noise characteristics of FM and PCM-FS systems, each of which, together with an SSB system, can be considered to be one of the typical systems for transmitting analog information signals. As results of these studies many curves for FSK, FM, and PCM-FS systems are given as materials for designing communication systems. SSB, FM, and PCM-FS systems are compared with each other with the use of several criteria of system comparison.

The main results obtained in this paper, besides the design materials, are summarized as follows:

1. The required intrinsic SNR for an assigned value of symbol error rate in multiple FSK systems can be reduced by increasing the number of frequencies in the keying. By using 3 or 4 frequencies the required intrinsic SNR can be reduced beyond that in a binary FSK system without requiring a wider overall bandwidth than in the binary system.

2. When the transmission rate is smaller than the channel capacity in multiple FSK systems, the element error rate can be made smaller than any assigned fixed value, no matter how small it is, by increasing the number of frequencies in the keying.

3. A conceptual FM demodulator of a new type, called the band-dividing FM demodulator, is introduced in order to study the essential signal-to-noise characteristics of an FM system. By frequency-modulating the carrier with sampled values from the original information signal to be transmitted and by demodulating the modulated wave with a band-dividing FM demodulator, it is possible to improve the threshold of an FM system beyond the threshold of an FM system having a conventional FM demodulator consisting of an amplitude limiter, a frequency discriminator, and a low-pass filter.

4. In band-dividing FM systems, as in conventional FM systems, the value of the intrinsic SNR at the threshold is not constant but increases as the modulation index (or deviation ratio) of the system increases.

5. It is suggested that the threshold in a frequency-lock or phase-lock FM demodulator cannot be improved beyond that in a band-dividing FM demodulator.

6. In PCM-FS systems the maximum output SNR depends only on the number of quantizing levels and not on the base in the coding, whereas the values of the intrinsic SNR at the threshold can be reduced by increasing the base when the number of quantizing levels is kept constant. By making the base in the coding equal to 3 or 4 the threshold can be improved beyond that in a binary PCM-FS system without requiring a wider overall bandwidth than in the binary system.

7. In PCM-FS systems the value of the intrinsic SNR at the threshold increases with the number of quantizing levels when either the base in the coding or the number of elements for each sample is kept constant.

8. When the assigned value of the output SNR is equal to or smaller than 50 decibels, an FM system having a band-dividing demodulator can be absolutely better than a PCM-FS system, i.e., both the required intrinsic SNR (or signal power) and the required

bandwidth in the former system are smaller than those in the latter.

9. When the intrinsic SNR of 30 decibels or larger can be used, both the FM and PCM-FS systems operate in the above-the-threshold region, and a PCM-FS system requires a narrower bandwidth than an FM system to obtain relatively large values of the output SNR (55 decibels or larger). Above the threshold, as the overall bandwidth is increased, the output SNR expressed in decibels varies as a logarithm of the bandwidth in an FM system, while it varies as the bandwidth in a PCM-FS system.

10. When the transmitter power is required to be a minimum, an FM system having a band-dividing demodulator is always better than a PCM-FS system. Even if a conventional demodulator is used in an FM system, it is better than a binary PCM-FS system unless the assigned output SNR is larger than approximately 55 decibels.

11. From the standpoint of the channel capacity an SSB system is always better than FM or PCM-FS systems.

The above studies indicate that the comparison of the communication systems can give different results according to the criterion of comparison adopted. It must be stressed, therefore, that the selection of the system or the determination of the system parameters depends on the condition required in each case.

The selection or the design of the communication system does not depend only on the signal-to-noise characteristics, although it is very important. As has been shown, a PCM-FS system is not generally superior to other systems insofar as the signal-to-noise characteristic alone is concerned. This fact, however, does not deny the possible advantages of a PCM-FS system, but also shows us the necessity of considering other characteristics. As is well known, a PCM-FS system is suited for a relay system with a long chain of repeaters because of the regeneration of the signal in each repeater [Oliver, et.al., 1948] .

In a PCM-FS system there exists a possibility of using an error-correcting code. It is very complicated, although not difficult in principle, to study the signal-to-noise characteristics of a PCM-FS system using an error-correcting code, because, by adding the parity-check digits to the information digits, we must increase the keying rate to keep the same information transmission rate, and a higher keying rate requires a wider overall bandwidth. These characteristics should be studied in the near future.

From the studies on the signal-to-noise characteristics of FM systems in this paper an interesting problem is raised. This problem is to study experimentally the mechanism of the loss-of-lock in a frequency-lock or a phase-lock FM demodulator in comparison with the mechanism of the mis-selection of the signal channel in a band-dividing FM demodulator. This problem is not only interesting but also important in practice.

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REFERENCES

Akima, Hiroshi, Signal-to-noise characteristics of some typical systems and their comparisons, Proceedings of the Navy Research and Development Clinic 1961.

Akima, Hiroshi, Theoretical studies on signal-to-noise characteristics of an FM system, (To be published).

Armstrong, Edwin H., A method of reducing disturbances in radio signaling by a system of frequency modulation, Proc. IRE 24, 689-740 (May 1936).

Battail, G., Determination approximative de la position extreme du seuil de reception en modulation de frequence, IRE Trans. on Information Theory IT-8, 108-121 (Sept. 1962).

Beard, J. V., and A. J. Wheeldon, A comparison between alternative HF telegraph systems, Point-to-Point Telecommunications 4, 20-48 (June 1960).

Billings, A. R., The rate of transmission of information in pulse-code-modulation systems, Proc. IEE 105-C, 444-447 (Sept. 1958).

Chaffee, J. G., The application of negative feedback to frequency-modulated systems, Proc. IRE 27, 317-331 (May 1939).

Choate, Robert L., Analysis of a phase-modulation communications system, IRE Trans. on Communications Systems CS-8, 221-227 (Dec. 1960).

Crosby, Murray G., Frequency modulation noise characteristics, Proc. IRE 25, 472-514 (April 1937).

de Jager, F., and J. A. Greefkes, Frena, a system of speech transmission at high noise levels, Philips Tech. Rev. 19, 73-83 (Oct. 1957).

Enloe, L. H., Decreasing the threshold in FM by frequency feedback, Proc. IRE 50, 18-30 (January 1962).

Gilchriest, C. E., Application of the phase-locked loop to telemetry as a discriminator or tracking filter, IRE Trans. on Telemetry and Remote Control TRC-4, 20-35 (June 1958).

Goodall, W. M., Telephony by pulse code modulation, Bell Sys. Tech. Jour. 26, 395-409 (July 1947).

Helstrom, C. W., The comparison of digital communication systems, IRE Trans. on Communications Systems CS-8, 141-150 (Sept. 1960).

Jaffe, R., and E. Rechtin, Design and performance of phase-lock circuits capable of near-optimum performance over a wide range of input signal and noise levels, IRE Trans. on Information Theory IT-1, 66-76 (Mar. 1955).

Jelonek, Z., A comparison of transmission systems, Symposium on Applications of Communication Theory, London, England (Sept. 1952).

Jordan, D. B., H. Greenberg, E. E. Eldredge, and W. Serniuk, Multiple frequency shift teletype systems, Proc. IRE 43, 1647-1655 (Nov. 1955).

Lehan, Frank W., and Robert J. Parks, Optimum demodulation, 1953 IRE National Convention Record, part 8, 101-103 (Mar. 1953).

Lehan, Frank W., Telemetering and information theory, IRE Trans. on Telemetry and Remote Control, PGRTRC-2, 15-19 (Nov. 1954).

Lieberman, Gilbert, Quantization in coherent and quadrature reception of orthogonal signals, RCA Rev. 22, 461-486 (Sept. 1961).

Margolis, Stephen G., The response of a phase-locked loop to a sinusoid plus noise, IRE Trans. on Information Theory IT-3, 136-142 (June 1957).

Martin, Benn D., Threshold improvement in an FM subcarrier system, IRE Trans. on Space Electronics and Telemetry SET-6, 25-33 (Mar. 1960).

Morita, M., and S. Ito, High sensitivity receiving system for frequency modulated wave, 1960 IRE International Convention Record, part 5, 228-237 (Mar. 1960).

National Bureau of Standards, Tables of Normal Probability Functions, U. S. Government Printing Office, Washington, D. C. (June 1953).

Oliver, B. M., J. R. Pierce, and C. E. Shannon, The philosophy of PCM, Proc. IRE 36, 1324-1331 (Nov. 1948).

Reeves, A. H., U. S. Patent No. 2,272,070 (Feb. 3, 1942); also French Patent No. 852,183 (Oct. 23, 1939).

Reiger, Siegfried, Error rates in data transmission, Proc. IRE 46, 919-920 (May 1958).

Robin, H. K., and T. L. Murray, Electronic multiplex 32-tone teleprinter system, National Research Development Corporation Bulletin, England, 13, 11-17 (October 1958).

Ruthroff, Clyde L., FM demodulators with negative feedback, Bell Sys. Tech. Jour., 40, 1149-1156 (July 1961).

Shannon, C. E., A mathematical theory of communication, Bell Sys. Tech. Jour., 27, 379-423 (July 1948), and 27, 623-656 (Oct. 1948).

Spilker, Jr., J. J., Threshold comparison of phase-lock, frequency-lock and maximum-likelihood types of FM discriminators, 1961 WESCON Convention Record, no. 14/2 (August 1961).

Stumpers, F.L.H.M., Theory of frequency-modulation noise, Proc. IRE, 36, 1081-1092 (Sept. 1948).

Turin, G. L., The asymptotic behavior of ideal M-ary systems, Proc. IRE, 47, 93-94 (Jan. 1959).

Viterbi, Andrew, Classification and evaluation of coherent synchronous sampled-data telemetry systems, IRE Trans. on Space Electronics and Telemetry, SET-8, 13-22 (Mar. 1962).

Weaver, Charles Sinclair, A new approach to the linear design and analysis of phase-locked loops, IRE Trans. on Space Electronics and Telemetry, SET-5, 166-178 (Dec. 1959).

APPENDIX A. AN APPROXIMATE METHOD OF CALCULATING THE ELEMENT ERROR RATE IN INCOHERENT MULTIPLE FSK SYSTEMS WHEN THE NUMBER OF FREQUENCIES IN THE KEYING IS VERY LARGE.

As given in the text the element error rate p_e in multiple FSK systems with incoherent detectors is given by

$$p_e = \int_0^{\infty} f(v, v_s) \left[1 - g^{N-1}(v) \right] dv, \tag{A-1}$$

where

$$f(v, v_s) = v \exp\left(- \frac{v^2 + v_s^2}{2} \right) I_0 (v_s v) \tag{A-2}$$

and

$$g(v) = 1 - \exp\left(- \frac{v^2}{2} \right). \tag{A-3}$$

When the number of frequencies in the keying N is much larger than one, an error rate of interest occurs at $v_s >> 1$, and the function $f(v, v_s)$ can be approximated by [Turin, 1959]

$$f(v, v_s) \cong \frac{1}{\sqrt{2\pi}} \exp\left(- \frac{(v - v_s)^2}{2} \right). \tag{A-4}$$

The function $g^{N-1}(v)$ is a monotonic increasing function of v from zero to unity as v varies from zero to infinity. It increases very rapidly somewhere depending on the value of N, and it increases very slowly elsewhere. We assume that $g^{N-1}(v)$ can be approximated by

$$h(v) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{v} \exp\left(- \frac{(u - a)^2}{2\sigma^2} \right) du, \tag{A-5}$$

where the two constants a and σ are so determined that, when $h(v) = 1/2$, $h(v)$ and $dh(v)/dv$ coincide with $g^{N-1}(v)$ and $dg^{N-1}(v)/dv$, respectively. From these assumptions we can obtain

$$a = \sqrt{2 \log_e \left(\frac{N-1}{\log_e 2} \right)} \tag{A-6}$$

and

$$\sigma = \sqrt{\frac{2}{\pi}} \; \frac{1 - \frac{\log_e 2}{N - 1}}{a \log_e 2} \; . \qquad \text{(A-7)}$$

The element error rate p_e is then approximated by

$$p_e \cong \frac{1}{2\pi\sigma} \int_{-\infty}^{\infty} \exp\left(-\frac{(v - v_s)^2}{2}\right) dv \int_{v}^{\infty} \exp\left(-\frac{(u - a)^2}{2^2}\right) du. \qquad \text{(A-8)}$$

If we transform the coordinate system (u, v) to (x, y) by the relations

$$\frac{u - a}{\sigma} = x \cos\theta + y \sin\theta \qquad \text{(A-9)}$$

and

$$v - v_s = - x \sin\theta + y \cos\theta , \qquad \text{(A-10)}$$

where

$$\theta = \cot^{-1} \sigma , \qquad \text{(A-11)}$$

the integrating range $u \geq v$ is transformed to

$$x \geq x_0 = \frac{v_s - a}{\sqrt{1 + \sigma^2}} , \qquad \text{(A-12)}$$

and the element error rate p_e can be given by

$$p_e \cong \frac{1}{2\pi} \int_{x_0}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{2}\right) dy = \frac{1}{\sqrt{2\pi}} \int_{x_0}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx. \qquad \text{(A-13)}$$

The values of x_0 can be determined for given values of N and v_s from (A-6), (A-7), and (A-12), and the values of the integral in (A-13) can be obtained from the tables of normal probability functions [National Bureau of Standards, 1953] .

In figure A-1 the relations between $R_c = v_s^2 / 2$ and p_e thus calculated approximately are compared with those calculated from (3) in the text. It is evident that the error due to the approximation lies within 0.2 decibels for N = 1024, and that the accuracy of the approximation is improved as N increases.

ACCURACY OF THE APPROXIMATION FOR EVALUATING THE ELEMENT ERROR RATE IN INCOHERENT FSK SYSTEMS

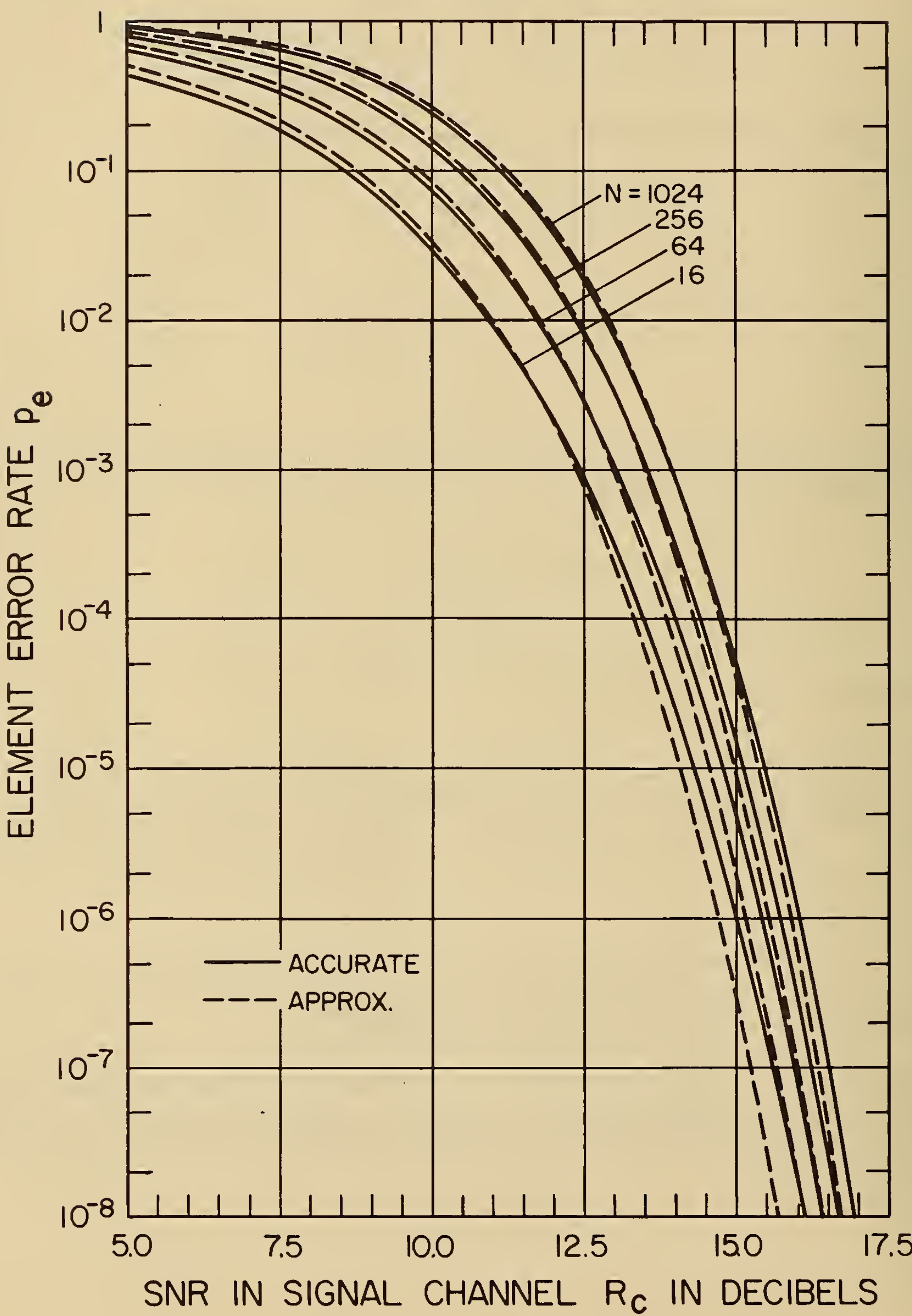

Figure A-1

APPENDIX B. THE OUTPUT SNR IN BAND-DIVIDING FM SYSTEMS

The output SNR R_{out} in band-dividing FM systems is given by

$$R_{out} = \frac{P_s}{P_{n1} + P_{n2}} , \tag{B-1}$$

where P_s is the output signal power, and P_{n1} and P_{n2} are the output noise power due to incoming noise into the signal channel and the output noise power due to mis-selection of the signal channel, respectively.

In order to calculate the output SNR the modulation index m is assumed to be an integer. In our demodulator the overall bandwidth B_a, the bandwidth of each channel B_c, and the number of channels N are given by

$$B_a = 2(m + 1) f_m = 2(m + 1) B_i, \tag{B-2}$$

$$B_c = 2f_m = 2B_i, \tag{B-3}$$

and

$$N = m + 1, \tag{B-4}$$

respectively, where f_m and B_i are the maximum frequency and the intrinsic bandwidth of the information signal, respectively. We also assume that the characteristics of the frequency-measuring circuit in each channel are linear within the frequency range from $(f_0 - B_a/2)$ to $(f_0 + B_a/2)$, and the output voltage of each circuit corresponding to the frequency of input wave of $(f_0 - B_a/2)$ and $(f_0 + B_a/2)$ are equal to $-N/2$ and $+N/2$, respectively, where f_0 is the center frequency of the modulated wave.

As will be shown later, the value of the channel SNR R_c at the threshold of this demodulator is larger than 10 decibels. Accordingly the output noise spectrum of the signal channel can be considered to be triangular [Crosby, 1937; Stumpers, 1948] and P_{n1} is given by

$$P_{n1} = \frac{1}{R_c} \int_0^{B_c/2} \left(\frac{f}{B_c} \right)^2 \frac{df}{B_c} = \frac{1}{24R_c} = \frac{1}{12R_i} . \tag{B-5}$$

If we denote the probability that the signal channel is correctly selected as the signal channel by q and the probability that any noise channel is selected by mistake as the signal channel by p, the probabilities p and q are related to the element error rate p_e in incoherent N-ary FSK systems by

$$(N - 1) p + q = p_e + q = 1. \tag{B-6}$$

When the ith channel ($i = 0, 1, 2, \ldots, N-1$) is occupied by the signal in some Nyquist interval, the expected value of i is given by

$$\bar{i} = \left(\sum_{j=0}^{N-1} j - i \right) p + iq = i + \left(\frac{N-1}{2} - i \right) Np. \qquad \text{(B-7)}$$

From (B-7) the relation

$$\bar{i} - \frac{N-1}{2} = (1 - Np)\left(i - \frac{N-1}{2} \right) \qquad \text{(B-8)}$$

can be obtained. As the $(N-1)/2$ th channel is the center channel, $\bar{i} - (N-1)/2$ is equal to the average output signal voltage when the ith channel is occupied by the signal. Equation (B-8) means that the modulation looks as if it is suppressed by a ratio equal to $(1 - Np)$. This is called the phenomenon of "modulation suppression". The output signal power for full modulation by a sinusoidal signal P_s, therefore, is given by

$$P_s = \frac{1}{2} \left(\frac{N-1}{2} \right)^2 (1 - Np)^2, \qquad \text{(B-9)}$$

because the output voltage $V_{s\ max}$ corresponding to the maximum frequency deviation $f_d = mf_m$ is equal to

$$V_{s\ max} = \frac{N}{2} \cdot \frac{m}{m+1} = \frac{N-1}{2} . \qquad \text{(B-10)}$$

Next the variance of the channel number can be calculated as follows.

$$\overline{(i - \bar{i})^2} = \overline{i^2} - \bar{i}^2 = \left[\left(\sum_{j=0}^{N-1} j^2 - i^2 \right) p + i^2 q \right] - \bar{i}^2$$

$$= \left[\frac{(N-1)(2N-1)}{6} - (N-1)\, i + i^2 \right] Np - \left(\frac{N-1}{2} - i \right)^2 N^2 p^2 . \qquad \text{(B-11)}$$

This is the output noise power due to mis-selection of the signal channel when the ith channel is occupied by the signal. It is clear from (B-11) that the output noise power depends on the value of i. If an equal probability of appearance of i is assumed, the output noise power due to mis-selection of the signal channel P_{n2} is given by

$$P_{n2} = \frac{1}{N} \sum_{i=0}^{N-1} \overline{(i - \bar{i})^2} = \frac{(N^2 - 1)(2 - Np)Np}{12} . \qquad \text{(B-12)}$$

From (B-1), (B-5), (B-9), and (B-12) the output SNR R_{out} is given by

$$R_{out} = \frac{3}{2}(N-1)^2 \frac{(1-Np)^2}{1/R_i + (N^2-1)(2-Np)Np} . \qquad \text{(B-13)}$$

It can also be written as

$$R_{out} = \frac{3}{2} m^2 R_i \frac{[1-(m+1)p]^2}{1+m(m+1)(m+2)[2-(m+1)p]pR_i} . \qquad \text{(B-14)}$$

(B-13) and (B-14) are given in the text as (29) and (18), respectively.

The relations between R_i and R_{out} are calculated numerically, and are shown as figure 9 in the text. We can see that the use of the triangular noise spectrum in (B-5) is supported by this figure, because the value of R_i at the threshold is larger than 13 db, which corresponds to R_c = 10 db.

APPENDIX C. THE OUTPUT SNR IN PCM-FS SYSTEMS

The output SNR R_{out} in PCM-FS systems is given by

$$R_{out} = \frac{P_s}{P_{n1} + P_{n2}}, \qquad \text{(C-1)}$$

where P_s is the output signal power, and P_{n1} and P_{n2} are the output noise power due to quantizing in the transmitter and the output noise power due to errors in the FSK transmission (or mis-selections of the signal channel), respectively.

In order to calculate the output SNR we take the spacing of the quantizing levels as the unit of output voltage.

The quantizing noise power P_{n1} is calculated as [Oliver, et al., 1948]

$$P_{n1} = \int_{-1/2}^{1/2} V^2 dV = 1/12. \qquad \text{(C-2)}$$

In N^n - PCM systems the quantizing level number of the transmitted signal ℓ_t can be expressed by

$$\ell_t = \sum_{k=1}^{n} i_k N^{k-1}, \qquad \text{(C-3)}$$

where i_k is zero or a positive integer from 1 to N - 1. The number of the total quantizing levels L is given by

$$L = N^n. \tag{C-4}$$

We denote, again, the probability that the signal channel is selected correctly as the signal channel by q and the probability that any noise channel is selected by mistake as the signal channel by p. The probabilities p and q are related to the element error rate p_e in N-ary FSK systems by

$$(N - 1) p + q = p_e + q = 1, \tag{C-5}$$

which is the same as (B-6).

If the errors occur in N-ary FSK transmission the quantizing level number of the received signal ℓ_r differs from that of the transmitted signal ℓ_t given in (C-3). The main problem in this Appendix is to calculate the average value and the variance of ℓ_r when errors occur in the FSK transmission.

We denote the difference between ℓ_r and ℓ_t by $\Delta\ell$, i.e.,

$$\Delta\ell = \ell_r - \ell_t. \tag{C-6}$$

We shall start with calculating the expected value of $\Delta\ell$ when the i_kth channel ($k = 1, 2, \ldots, n$) is occupied by the signal in some Nyquist interval.

If an error occurs in the kth element and j_k is received instead of i_k while the other (n - 1) elements are received correctly, the difference $\Delta\ell$ is given by

$$\Delta\ell = (j_k - i_k)N^{k-1}. \tag{C-7}$$

As the probability that such an error occurs is equal to pq^{n-1}, and as every value of j_k except i_k has the same value of probability of being received by mistake, the expected value of $\Delta\ell$ when an error occurs in the kth element while the other (n - 1) elements are received correctly is given by taking a summation of the product of pq^{n-1} and (C-7) over the values of j_k from 0 to N - 1 except $j_k = i_k$. This is calculated as

$$pq^{n-1}\left[\sum_{j_k=0}^{i_k-1} (\Delta\ell) + \sum_{j_k=i_k+1}^{N-1} (\Delta\ell)\right] = pq^{n-1}\sum_{j_k=0}^{N-1} (\Delta\ell)$$

$$= pq^{n-1}\left(\frac{N-1}{2} - i_k\right)N^k. \tag{C-8}$$

The expected value of $\Delta \ell$ due to a single error in any element is given by taking a summation of (C-8) over the value of k from 1 to n as

$$pq^{n-1} \sum_{k=1}^{n} \left(\frac{N-1}{2} - i_k \right) N^k = Npq^{n-1} \left(\frac{L-1}{2} - \ell_t \right). \tag{C-9}$$

We can extend the above result to the case where m elements out of n elements are received incorrectly. If errors occur in the specified m elements, say k_1th, k_2th, ..., and k_mth, and if a set of the specified values of j_{k_1}, j_{k_2}, ... , and j_{k_m} is received instead of the set of the correct values of i_{k_1}, ... , and i_{k_m} while the other (n - m) elements are received correctly, the difference $\Delta \ell$ is given by

$$\Delta \ell = \left(j_{k_1} - i_{k_1} \right) N^{k_1 - 1} + \left(j_{k_2} - i_{k_2} \right) N^{k_2 - 1} + \ldots . + \left(j_{k_m} - i_{k_m} \right) N^{k_m - 1}. \tag{C-10}$$

As the probability that such errors occur is $p^m q^{n-m}$, and as every value of j_{k_a} except i_{k_a} (a = 1, 2, ..., m) has the same value of probability of being received by mistake, the expected value of the difference $\Delta \ell$ due to the m errors in the k_1th, k_2th, ... , and k_mth elements is given by taking a summation of the product of $p^m q^{n-m}$ and (C-10) over the values of j_{k_a} from 0 to N - 1 except $j_{k_a} = i_{k_a}$. This is calculated as

$$p^m q^{n-m} \sum_{j_{k_1}} \ldots . \sum_{j_{k_m}} \left[\left(j_{k_1} - i_{k_1} \right) N^{k_1 - 1} + \ldots . + \left(j_{k_m} - i_{k_m} \right) N^{k_m - 1} \right]$$

$$= p^m q^{n-m} (N-1)^{m-1} \left[\left(\frac{N-1}{2} - i_{k_1} \right) N^{k_1} + \ldots + \left(\frac{N-1}{2} - i_{k_m} \right) N^{k_m} \right], \tag{C-11}$$

where each summation is taken over the value of j_{k_a} from 0 to (N - 1) except i_{k_a}. The expected value of the difference $\Delta \ell$ due to m errors in any combination of m elements is given by taking a summation over any combination of m elements. In this summation every value of k_a appears $\binom{n-1}{m-1}$ times, because the error in the k_ath element is accompanied by other (m - 1) errors in (m - 1) elements out of (n - 1) elements. The sum, therefore, is given by

$$p^{m} q^{n-m} (N - 1)^{m-1} \binom{n-1}{m-1} \sum_{k=1}^{n} \left(\frac{N-1}{2} - i_k \right) N^k$$

$$= p^{m} q^{n-m} (N - 1)^{m-1} \binom{n-1}{m-1} N \left(\frac{N^n - 1}{2} - \sum_{k=1}^{n} i_k N^{k-1} \right)$$

$$= \binom{n-1}{m-1} [(N - 1)p]^{m-1} q^{(n-1)-(m-1)} \left[Np \left(\frac{L-1}{2} - \ell_t \right) \right] . \tag{C-12}$$

When $m = 1$, (C-12) is identical to (C-9).

The expected value of the difference $\Delta \ell$ due to any combination of errors is given by taking a summation of (C-12) over m from 1 to n. From (C-5) the summation leads to

$$\overline{\Delta \ell} = \left[Np \left(\frac{L-1}{2} - \ell_t \right) \right] \sum_{m-1}^{n} \binom{n-1}{m-1} [(N - 1)p]^{m-1} q^{(n-1)-(m-1)}$$

$$= Np \left(\frac{L-1}{2} - \ell_t \right) \left[(N - 1)p + q \right]^{n-1}$$

$$= Np \left(\frac{L-1}{2} - \ell_t \right) . \tag{C-13}$$

From (C-13) the expected value of the level number of the received signal ℓ_r is given by

$$\overline{\ell_r} = \ell_t + Np \left(\frac{L-1}{2} - \ell_t \right) , \tag{C-14}$$

and the relation

$$\overline{\ell_r} - \frac{L-1}{2} = (1 - Np) \left(\ell_t - \frac{L-1}{2} \right) \tag{C-15}$$

is obtained from (C-14). As the $(L - 1)/2$ th level is the center, $\overline{\ell_r} - (L - 1)/2$ is equal to the average output signal voltage when the signal voltage $\ell_t - (L - 1)/2$ is transmitted. Equation (C-15) means that the ratio of "modulation suppression" is equal to $(1 - Np)$. The output signal power for full modulation by a sinusoidal signal P_s, therefore, is given by

$$P_s = \frac{1}{2} \left(\frac{L-1}{2} \right)^2 (1 - Np)^2 . \tag{C-16}$$

Similarly, we can calculate the variance of ℓ_r from the expected value of $(\Delta\ell)^2$ when errors occur in N-ary FSK transmission. If errors occur in the specified m elements, say k_1th, k_2th, ..., and k_mth, and if a set of the specified values of j_{k_1}, j_{k_2}, ..., and j_{k_m} is received instead of the set of the correct values of i_{k_1}, i_{k_2}, ..., and i_{k_m} while the other (n - m) elements are received correctly, the square of the difference $(\Delta\ell)^2$ is given by

$$(\Delta\ell)^2 = \left[\left(j_{k_1} - i_{k_1}\right) N^{k_1 - 1} + \ldots + \left(j_{k_m} - i_{k_m}\right) N^{k_m - 1}\right]^2 . \qquad \text{(C-17)}$$

As the probability that such errors occur is $p^m q^{n-m}$, and as every value of j_{k_a} except i_{k_a} (a = 1, 2, ..., m) has the same value of probability of being received by mistake, the expected value of the square of the difference $(\Delta\ell)^2$ due to the m errors in the k_1th, k_2th, ..., and k_mth elements is given by taking a summation of the product of $p^m q^{n-m}$ and (C-17) over the values of j_{k_a} from 0 to N - 1 except $j_{k_a} = i_{k_a}$. This is calculated as

$$p^m q^{n-m} \sum_{j_{k_1}} \cdots \sum_{j_{k_m}} \left[\left(j_{k_1} - i_{k_1}\right) N^{k_1 - 1} + \ldots + \left(j_{k_m} - i_{k_m}\right) N^{k_m - 1}\right]^2$$

$$= p^m q^{n-m} (N-1)^{m-1} \left\{ \sum_{a=1}^{m} \left[\sum_{j_{k_a}=0}^{N-1} \left(j_{k_a} - i_{k_a}\right)^2 N^{2(k_a - 1)} \right] \right\}$$

$$+ 2p^m q^{n-m} (N-1)^{m-2} \left\{ \sum_{a=1}^{m-1} \sum_{b=a+1}^{m} \left[\sum_{j_{k_a}=0}^{N-1} \sum_{j_{k_b}=0}^{N-1} \left(j_{k_a} - i_{k_a}\right)\left(j_{k_b} - i_{k_b}\right) N^{k_a + k_b - 2} \right] \right\}$$

$$= Np\left[(N-1)p\right]^{m-1} q^{(n-1)-(m-1)} \left\{ \sum_{a=1}^{m} \left[\frac{N^2 - 1}{12} + \left(\frac{N-1}{2} - i_{k_a}\right)^2 \right] N^{2(k_a - 1)} \right\}$$

$$+ 2(Np)^2 \left[(N-1)p\right]^{m-2} q^{(n-2)-(m-2)} \left\{ \sum_{a=1}^{m-1} \sum_{b=a+1}^{m} \left(\frac{N-1}{2} - i_{k_a}\right)\left(\frac{N-1}{2} - i_{k_b}\right) N^{k_a + k_b - 2} \right\}.$$

(C-18)

The expected value of the square of the difference due to m errors in any combination of m elements is given by taking a summation of (C-18) over any combination of m elements. In this summation every value of k_a appears $\binom{n-1}{m-1}$ times, because the error in the k_ath element is accompanied by other (m - 1) errors in (m - 1) elements out of (n - 1) elements, whereas every combination of k_a and k_b appears $\binom{n-2}{m-2}$ times, because the two errors in the k_ath and k_bth elements are accompanied by other (m -2) errors in (m - 2) elements out of (n - 2) elements. From (C-4) the summation leads to

$$Np\binom{n-1}{m-1}\left[(N-1)p\right]^{m-1} q^{(n-1)-(m-1)}\left[\frac{L^2-1}{12}+\sum_{k=1}^{n}\left(\frac{N-1}{2}-i_k\right)^2 N^{2(k-1)}\right]$$

$$+\ 2(Np)^2\binom{n-2}{m-2}\left[(N-1)p\right]^{m-2} q^{(n-2)-(m-2)}$$

$$\times\ \left[\sum_{k=1}^{n-1}\sum_{h=k+1}^{n}\left(\frac{N-1}{2}-i_k\right)\left(\frac{N-1}{2}-i_h\right)N^{k+h-2}\right]. \qquad \text{(C-19)}$$

The expected value of the square of the difference due to any combination of errors $\overline{(\Delta \ell)^2}$ is given by taking a summation of (C-19) over m from 1 to n. From (C-5) the summation leads to

$$\overline{(\Delta \ell)^2} = Np\left\{\frac{L^2-1}{12}+\sum_{k=1}^{n}\left[\left(\frac{N-1}{2}-i_k\right)N^{k-1}\right]^2\right\}$$

$$+\ 2(Np)^2\left\{\sum_{k=1}^{n-1}\sum_{h=k+1}^{n}\left[\left(\frac{N-1}{2}-i_k\right)N^{k-1}\right]\left[\left(\frac{N-1}{2}-i_h\right)N^{h-1}\right]\right\}. \qquad \text{(C-20)}$$

The variance of the level number in the received signal ℓ_r is related to $\overline{(\Delta \ell)^2}$ by

$$\overline{(\ell_r - \overline{\ell_r})^2} = \overline{(\Delta \ell)^2} - \overline{(\Delta \ell)}^2. \qquad \text{(C-21)}$$

The last term $\overline{(\Delta \ell)}^2$ in (C-21) can be expressed by

$$\overline{(\Delta \ell)}^2 = (Np)^2 \left(\frac{L-1}{2} - \sum_{k=1}^{n} i_k N^{k-1} \right)^2$$

$$= (Np)^2 \left[\sum_{k=1}^{n} \left(\frac{N-1}{2} - i_k \right) N^{k-1} \right]^2$$

$$= (Np)^2 \left\{ \sum_{k=1}^{n} \left[\left(\frac{N-1}{2} - i_k \right) N^{k-1} \right]^2 \right\}$$

$$+ 2(Np)^2 \left\{ \sum_{k=1}^{n-1} \sum_{h=k+1}^{n} \left[\left(\frac{N-1}{2} - i_k \right) N^{k-1} \right] \left[\left(\frac{N-1}{2} - i_h \right) N^{h-1} \right] \right\} \tag{C-22}$$

from (C-13). The variance of ℓ_r, therefore, is given by

$$\overline{(\ell_r - \overline{\ell_r})^2} = \frac{(L^2 - 1)Np}{12} + (1 - Np)Np \left\{ \sum_{k=1}^{n} \left[\left(\frac{N-1}{2} - i_k \right) N^{k-1} \right]^2 \right\}. \tag{C-23}$$

This is the output noise power due to errors in the FSK transmission. It is clear from (C-23) that the output noise power depends on the values of i_k. If an equal probability of appearance of each i_k is assumed, the output noise power due to errors is given by taking an average over every value of i_k as

$$P_{n2} = \frac{(L^2 - 1)Np}{12} + (1 - Np)Np \left\{ \sum_{k=1}^{n} \frac{1}{N} \left[\sum_{i_k=0}^{N-1} \left(\frac{N-1}{2} - i_k \right)^2 N^{2(k-1)} \right] \right\}$$

$$= \frac{(L^2 - 1)Np}{12} + (1 - Np)\, Np \left[\sum_{k=1}^{n} N^{2(k-1)} \right] \frac{1}{N} \left[\sum_{i=0}^{N-1} \left(\frac{N-1}{2} - i \right)^2 \right]$$

$$= \frac{(L^2 - 1)\,(2 - Np)\, Np}{12}. \tag{C-24}$$

From (C-1), (C-2), (C-16), and (C-24) the output SNR R_{out} is given by

$$R_{out} = \frac{3}{2} (L - 1)^2 \, \frac{(1 - Np)^2}{1 + (L^2 - 1)\,(2 - Np)\, Np}. \tag{C-25}$$

This result is given as (25) in the text.

LIST OF FIGURES

LIST OF SYMBOLS

(The symbols used only in Appendices are not listed here.)

B_a	Overall bandwidth of the system.
B_c	Channel bandwidth, which is the bandwidth of each channel in the demodulator.
B_i	Intrinsic bandwidth of the information signal. In the transmission of digital information signal it is equal to the reciprocal of the unit duration in a binary system. In the transmission of analog information signal it is equal to the maximum frequency of the information signal to be transmitted.
C	Channel capacity.
e	Base of the natural system of logarithms = 2.71828 ...
f_d	Maximum frequency deviation in FM systems.
f_m	Maximum frequency of the analog information signal to be transmitted.
$I_0(x)$	Modified Bessel function of the 1st kind of the 0th order.
k	A positive integer.
L	Number of symbols (or characters) in FSK systems; and number of quantizing levels in PCM-FS systems.
m	Modulation index (or deviation ratio) in FM systems.
n	Number of elements for each symbol (or character) in FSK systems; and number of elements for each sample in PCM-FS systems.
N	Number of frequencies in the keying in FSK systems; number of channel in band-dividing FM demodulators; and the base in the coding in PCM-FS systems.
$\binom{N}{k}$	Number of combinations of k out of N.
p	Probability that any noise channel is selected as the signal channel by mistake in band-dividing FM and PCM-FS systems.
p_e	Element error rate in FSK systems.
p_s	Symbol (or character) error rate in FSK systems.
R_a	Overall SNR, which is the ratio of the incoming signal power to the incoming noise power contained in a band of width B_a.
R_c	Channel SNR, which is the ratio of the incoming signal power to the incoming noise power contained in a band of width B_c.
R_i	Intrinsic SNR, which is the ratio of the incoming signal power to the incoming noise power contained in a band of width B_i.
R_{out}	Output SNR, which is the ratio of the output signal power to the output noise power.
u	A real variable.
v	A real variable.
v_s	Normalized amplitude of the incoming signal voltage with the effective value of the noise voltage in each channel as a unit.
π	The ratio of the circumference of a circle to its diameter = 3.14159...

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U. S. DEPARTMENT OF COMMERCE
Luther H. Hodges, *Secretary*

NATIONAL BUREAU OF STANDARDS
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